Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[5 points]** Short answer: what is the maximum size of a list of distinct integers with no increasing sublist of size 10 and no decreasing sublist of size 7?

$(10-1)\cdot(7-1) = 9\cdot 6 = \boxed{54}$.

Note: 55 is min size of a list in which an increasing sublist of size 10 or decreasing sublist of size 7 is forced

2. **[10 points]** Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Prove that if T is a subset of S of size 5, then T contains a pair of integers that sum to 9.

Consider the partition of S into parts $X_1, \ldots, X_4$ with $X_1 = \{1, 8\}$, $X_2 = \{2, 7\}$, $X_3 = \{3, 6\}$, and $X_4 = \{4, 5\}$. Since $T \subseteq S$ with $|T| \geq 5$, there exist distinct elements $a, b \in T$ that belong to the same part X_i by the pigeonhole principle. Since the two elements in X_i sum to 9, we have $a + b = 9$ as required. ∎

3. **[10 points]** Let $A = \{1, \ldots, n\}$, $B = \{n+1, \ldots, 2n\}$, and $C = \{2n+1, \ldots, 3n\}$. Prove that if $n \geq 15$, then there exists $a_1, a_2 \in A$, $b_1, b_2 \in B$, and $c_1, c_2 \in C$ such that $(a_1, b_1, c_1) \neq (a_2, b_2, c_2)$ and $a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2$.

Let $X = \{(a, b, c) : a \in A, b \in B, c \in C\} = A \times B \times C$, and define the function $f : X \to \mathbb{N}$ by $f(a, b, c) = a^2 + b^2 + c^2$. For all $(a, b, c) \in X$, we have

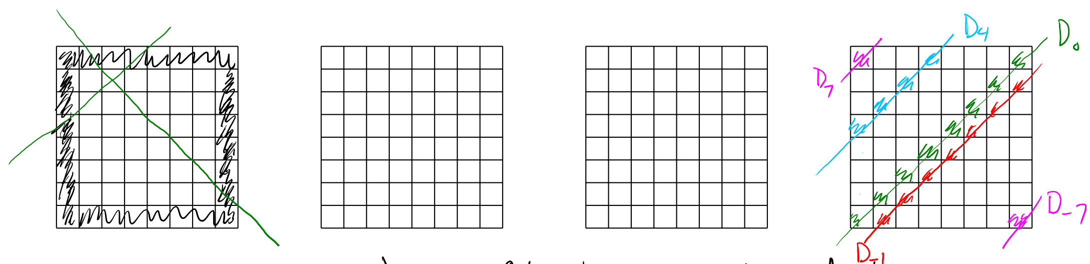
$$5n^2 < 1^2 + n^2 + (2n)^2 < a^2 + b^2 + c^2 = f(a, b, c) = a^2 + b^2 + c^2 \leq n^2 + (2n)^2 + (3n)^2 = 14n^2.$$

Hence the range of f is contained in $\{5n^2 + 1, \ldots, 14n^2\}$, a set of integers of size $9n^2$.

Note that $|X| = |A| \cdot |B| \cdot |C| = n^3 > 9n^2$ since $n \geq 15$, it follows from the pigeonhole principle that there exist distinct $(a_1, b_1, c_1), (a_2, b_2, c_2) \in X$ such that $f(a_1, b_1, c_1) = f(a_2, b_2, c_2)$. This implies $(a_1, b_1, c_1) \neq (a_2, b_2, c_2)$ and $a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2$. ∎

4. **[2 parts, 10 points each]** In a checkerboard, a *diagonal* is a maximal set of cells whose centers are on a line with a slope of 1 or −1. Two examples of diagonals follow.

(a) Show that it is possible to mark 28 cells of the 8×8 checkerboard such that every diagonal contains at most 2 marked cells. Four boards are given below for your convenience. Clearly indicate your final solution.



Mark the cells on the boundary. Clearly every diagonal has at most 2 marked cells.

(b) Prove that no matter how 29 cells are marked, some diagonal contains at least 3 marked cells.

Pf. Index the cells by the set $\{(x,y): x,y \in \{1,\dots,8\}\}$ and for $-7 \le k \le 7$, let $D_k = \{(x,y): y-x=k\}$, so that D_k is the set of cells in the diagonal whose centers are on a line with slope 1 and y-intercept k. Note that $\{D_{-7},\dots,D_0,\dots,D_7\}$ is a partition of the grid and $|D_k| = 8-|k|$. Let S be a set of marked cells in which each D_k has at most 2 marked cells. Since $|D_{-7}| = |D_7| = 1$, we compute

$$|S| = \sum_{k=-7}^{7} |D_k \cap S| = |D_{-7}\cap S| + |D_7 \cap S| + \sum_{k=-6}^{6} |D_k \cap S| \le 1 + 1 + \sum_{k=-6}^{6} 2$$

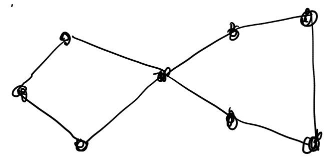
$$= 1 + 1 + 2\cdot 13 = 28.$$

It follows that if $|S| \ge 29$, then for some diagonal D_k, we have that $|S \cap D_k| \ge 3$. ☒

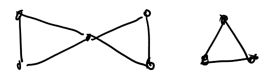
5. **[2 parts, 10 points each]** Graphs and degrees.

   (a) Construct an 8-vertex graph in which one vertex has degree 4 and the rest of the vertices have degree 2.

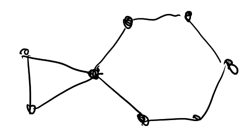
Soln 1:



Soln 2:



Soln 3:

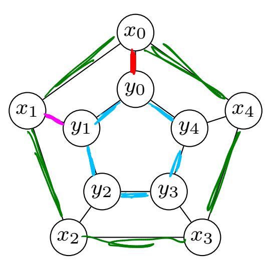


Three possible answers

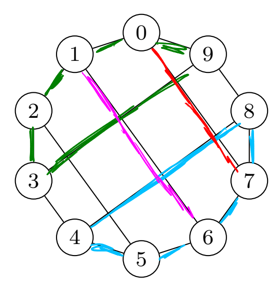
   (b) Prove that there is no 8-vertex bipartite graph in which one vertex has degree 4 and the rest of the vertices have degree 2. (Hint: suppose for a contradiction that G is such a bipartite graph with parts X and Y. What can you say about $\sum_{v\in V(G)} d(v)$ and $\sum_{v\in X} d(v)$?)

Suppose for a contradiction that G is such a graph with parts X and Y. By the degree-sum formula, $2|E(G)| = \sum_{v\in V(G)} d(v) = 4 + 7\cdot 2 = 18$, implying $|E(G)| = \frac{18}{2} = 9$. Since G is bipartite, each edge has exactly one endpoint in X, implying $\sum_{v\in X} d(v) = |E(G)| = 9$. But every degree in G is even, and so $\sum_{v\in X} d(v)$ is even, a contradiction. It follows that there is no such bipartite graph. ∎

6. **[10 points]** Determine if G and H are isomorphic. If they are isomorphic, then give an isomorphism. If not, then give a property that distinguishes the graphs.



G

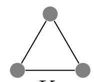


H

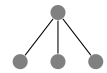
These graphs are $\boxed{\text{isomorphic}}$, for example via the following isomorphism:

$V(G)$	x_0	x_1	x_2	x_3	x_4	y_0	y_1	y_2	y_3	y_4
$V(H)$	0	1	2	3	9	7	6	5	4	8

7. **[25 points]** Recall that K_3 is the triangle and $K_{1,3}$ is the complete bipartite graph with parts of sizes 1 and 3. Show that $r(K_3, K_{1,3}) = 7$. Be sure to show both that $r(K_3, K_{1,3}) > 6$ and $r(K_3, K_{1,3}) \leq 7$.

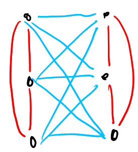


K_3



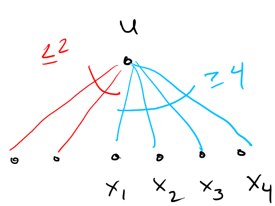
$K_{1,3}$

$\underline{r(K_3, K_{1,3}) > 6}$: We give a {blue, red}-edge-coloring of K_6 that avoids a blue copy of K_3 and avoids a red copy of $K_{1,3}$.



The blue subgraph is $K_{3,3}$ and hence bipartite, so the blue subgraph contains no copy of K_3. Also, the red subgraph is two disjoint copies of K_3 and hence has maximum degree 2, implying the red subgraph contains no copy of $K_{1,3}$.

$\underline{r(K_3, K_{1,3}) \leq 7}$: Let G be a {blue, red}-edge-coloring of K_7. We show G contains a blue copy of K_3 or a red copy of $K_{1,3}$. Let $u \in V(G)$. If u has red-degree at least 3, then u together with 3 red neighbors gives a copy of $K_{1,3}$ in red.



Hence u has at least 4 blue neighbors x_1, x_2, x_3, x_4. If some pair in $\{x_1, \ldots, x_4\}$ is blue, then this edge extends via u to give a blue copy of K_3. Hence we may assume every pair in $\{x_1, \ldots, x_4\}$ is red, giving a red copy of K_4. Since K_4 contains $K_{1,3}$ as a subgraph, we also obtain a red copy of $K_{1,3}$. In all cases, G has a blue copy of K_3 or a red copy of $K_{1,3}$, and so $r(K_3, K_{1,3}) \leq 7$. ∎