Name: ______________________________

Directions: Show all work. No credit for answers without work.

1. Prove the following by induction or the no minimum counter-example technique.

   (a) **[15 points]** Recall that $\hat{F}_0 = \hat{F}_1 = 1$ and $\hat{F}_n = \hat{F}_{n-1} + \hat{F}_{n-2}$ for $n \geq 2$. Prove that $\sum_{k=0}^{n} \hat{F}_k = \hat{F}_{n+2} - 1$ for $n \geq 0$.

   (b) **[15 points]** Prove that for $n \geq 0$, we have $\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = 1 - \frac{1}{(n+1)^2}$.

2. **[20 points]** Let n be a positive odd integer, and let G_n be the $n \times n$ grid with n^2 cells. For $0 \leq x, y \leq n-1$, let (x, y) denote the cell of G_n in column x and row y. Let $G_n - (x, y)$ denote G_n with the cell (x, y) removed. Prove that if $x + y$ is odd, then $G_n - (x, y)$ **cannot** be tiled with dominoes.

$(0,4)$	$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$
$(0,3)$	$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$
$(0,2)$	$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$
$(0,1)$	$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$
$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(4,0)$

The grid G_5

$(0,4)$	$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$
$(0,3)$	$(1,3)$		$(3,3)$	$(4,3)$
$(0,2)$	$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$
$(0,1)$	$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$
$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(4,0)$

The grid $G_5 - (2,3)$

3. Use the characteristic equation method to solve the following.

   (a) [**15 points**] Find the general solution to the recurrence $a_n = 3a_{n-2} + 2a_{n-3}$ for $n \geq 3$.

   (b) [**10 points**] Find a closed form formula for a_n with base cases $a_0 = a_1 = 4$ and $a_2 = 15$.

4. Let n be a positive integer.

   (a) **[10 points]** Give an example of a subset A of $\{1, \ldots, 3n\}$ of size $2n$ such that there is no pair $x, y \in A$ with $y - x = n$.

   (b) **[15 points]** Prove that if $A \subseteq \{1, \ldots, 3n\}$ and $|A| \geq 2n + 1$, then A contains a pair $x, y \in A$ with $y - x = n$.