Name: Solutions

Directions: Show all work. No credit for answers without work.

1. Prove the following by induction or the no minimum counter-example technique.

   (a) **[15 points]** Recall that $\hat{F}_0 = \hat{F}_1 = 1$ and $\hat{F}_n = \hat{F}_{n-1} + \hat{F}_{n-2}$ for $n \geq 2$. Prove that $\sum_{k=0}^{n} \hat{F}_k = \hat{F}_{n+2} - 1$ for $n \geq 0$.

By induction on n. Basis step: If $n=0$, then $\sum_{k=0}^{n} \hat{F}_k = F_0 = 1$. Also, $\hat{F}_{n+2} - 1 = \hat{F}_2 - 1 = 2 - 1 = 1$, and so the equality holds when $n=0$.

Inductive Step: Suppose $n \geq 1$. By the inductive hypothesis, we have $\sum_{k=0}^{n-1} \hat{F}_k = \hat{F}_{n+1} - 1$.

Adding $\hat{F}_n$ to both sides gives $\sum_{k=0}^{n} \hat{F}_k = (\hat{F}_{n+1} - 1) + \hat{F}_n = \hat{F}_{n+1} + \hat{F}_n - 1 = \hat{F}_{n+2} - 1$.

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   (b) **[15 points]** Prove that for $n \geq 0$, we have $\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = 1 - \frac{1}{(n+1)^2}$.

Pf: By induction on n. Basis Step: Suppose $n=0$. Then $\sum_{k=1}^{0} \frac{2k+1}{k^2(k+1)^2}$ is the empty sum with value 0. Also, $1 - \frac{1}{(0+1)^2} = 1 - 1 = 0$. So the identity holds when $n=0$.

Inductive Step: Suppose $n \geq 1$. By the inductive hypothesis, we have

$$\sum_{k=1}^{n-1} \frac{2k+1}{k^2(k+1)^2} = 1 - \frac{1}{n^2}. \quad (*)$$

Adding $\frac{2n+1}{n^2(n+1)^2}$ to both sides of $(*)$ gives

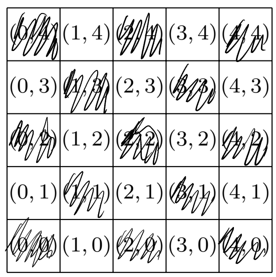
$$\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = 1 - \frac{1}{n^2} + \frac{2n+1}{n^2(n+1)^2} = 1 - \left(\frac{(n+1)^2}{n^2(n+1)^2} - \frac{2n+1}{n^2(n+1)^2}\right)$$

$$= 1 - \left(\frac{(n+1)^2 - (2n+1)}{n^2(n+1)^2}\right) = 1 - \left(\frac{(n^2+2n+1) - (2n+1)}{n^2(n+1)^2}\right)$$

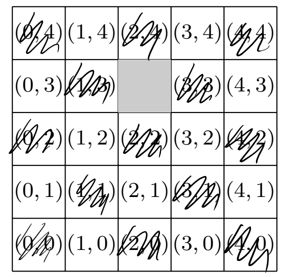
$$= 1 - \frac{n^2}{n^2(n+1)^2} = 1 - \frac{1}{(n+1)^2}.$$

∎

2. **[20 points]** Let n be a positive odd integer, and let G_n be the $n \times n$ grid with n^2 cells. For $0 \le x, y \le n-1$, let (x, y) denote the cell of G_n in column x and row y. Let $G_n - (x, y)$ denote G_n with the cell (x, y) removed. Prove that if $x + y$ is odd, then $G_n - (x, y)$ **cannot** be tiled with dominoes.



The grid G_5

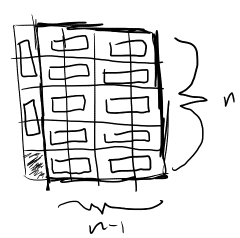


The grid $G_5 - (2, 3)$

Pf. Color G_n so that (i,j) is black if $i+j$ is even and (i,j) is white if $i+j$ is odd. This creates a checkerboard pattern. Suppose for a contradiction that T is a domino tiling of $G_n - (x,y)$. Since each domino covers one black cell and one white cell, it follows that $G_n - (x,y)$ has the same number of black cells as white cells.

Next, we claim that $G_n - (0,0)$ has the same number of black cells as white cells, and so there are $\frac{n^2-1}{2}$ cells of each color. This follows from tiling the $(n-1) \times n$ subgrid $\{(x,y) : 1 \le x \le n-1, 0 \le y \le n-1\}$ with horizontal dominos and the strip $\{(0, y) : 1 \le y \le n-1\}$ with vertical dominos, as follows:

$G - (0,0)$



Since $(0,0)$ is black, G_n has one more black cell than white. Since $x+y$ is odd, (x,y) is white. So $G_n - (x,y)$ has two more black cells than white. This contradicts that $G_n - (x,y)$ has the same number of black and white cells. $\blacksquare$

3. Use the characteristic equation method to solve the following.

(a) **[15 points]** Find the general solution to the recurrence $a_n = 3a_{n-2} + 2a_{n-3}$ for $n \geq 3$.

Char Eqn: $x^3 = 3x + 2$

$x^3 - 3x - 2 = 0$

Note: $x = 2$ is a root.

$$\begin{array}{r} x^2 + 2x + 1 \\ x-2 \overline{\big) x^3 \quad -3x - 2} \\ \underline{x^3 - 2x^2 \qquad} \\ 2x^2 - 3x - 2 \\ \underline{-(2x^2 - 4x)} \\ x - 2 \\ \underline{x - 2} \\ 0 \end{array}$$

So $x^3 - 3x - 2 = (x-2)(x^2+2x+1)$
$= (x-2)(x+1)^2$

Roots: $r_1 = 2$, mult 1, $r_2 = -1$, mult 2.

Gen Soln: $a_n = p_1 \cdot 2^n + p_2 \cdot (-1)^n$, where p_1 has deg 0 and p_2 has degree 1.

So

$$\boxed{a_n = \alpha \cdot 2^n + (\beta n + \gamma) \cdot (-1)^n}$$

(b) **[10 points]** Find a closed form formula for a_n with base cases $a_0 = a_1 = 4$ and $a_2 = 15$.

Impose base cases:

$\underline{a_0 = 4}$: $\alpha \cdot 2^0 + (\beta \cdot 0 + \gamma) \cdot (-1)^0 = 4$

$\alpha + \gamma = 4$

$\underline{a_1 = 4}$ $\alpha \cdot 2^1 + (\beta \cdot 1 + \gamma)(-1)^1 = 4$

$2\alpha - \beta - \gamma = 4$

$\underline{a_2 = 15}$: $\alpha \cdot 2^2 + (\beta \cdot 2 + \gamma)(-1)^2 = 15$

$4\alpha + 2\beta + \gamma = 15$

Matrix/Lin Alg Soln:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & -1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 15 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 2 & -1 & -1 & 4 \\ 4 & 2 & 1 & 15 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & -1 & -3 & -4 \\ 0 & 2 & -3 & -1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & -3 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -9 & -9 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\alpha = 3$, $\beta = 1$, $\gamma = 1$

$$\boxed{a_n = 3 \cdot 2^n + (n+1) \cdot (-1)^n}$$

4. Let n be a positive integer.

(a) **[10 points]** Give an example of a subset A of $\{1, \ldots, 3n\}$ of size $2n$ such that there is no pair $x, y \in A$ with $y - x = n$.

Let $A_1 = \{1, \ldots, n\}$ and $A_2 = \{2n+1, \ldots, 3n\}$, and let $A = A_1 \cup A_2$. Let $x, y \in A$. If $x \geq y$, then $y - x \leq 0$ and so $y - x \neq n$ since $n > 0$. So assume $y > x$. If $y \in A_2$ and $x \in A_1$, then $y - x \geq (2n+1) - n = n+1$, and so $y - x \neq n$. If $x, y \in A_1$, then $y - x \leq n-1$ and so $y - x \neq n$. Finally, if $x, y \in A_2$, then $y - x \leq 3n - (2n+1) = n-1$ and again $y - x \neq n$.

In all cases, $y - x \neq n$. Also $|A| = |A_1| + |A_2| = n + n = 2n$. $\boxtimes$

(b) **[15 points]** Prove that if $A \subseteq \{1, \ldots, 3n\}$ and $|A| \geq 2n+1$, then A contains a pair $x, y \in A$ with $y - x = n$.

Partition $\{1, \ldots, 3n\}$ into n parts $X_1, \ldots, X_n$ of size 3 where $X_t = \{t, t+n, t+2n\}$ for $1 \leq t \leq n$. Since A has more than $2n$ elements and each belongs to one of the n parts $X_1, \ldots, X_n$, it follows from the pigeonhole principle that some X_t contains at least 3 elements from A, which implies $A \cap X_t = \{t, t+n, t+2n\}$. We may now take $y = t+n$ and $x = t$, and so $y - x = (t+n) - t = n$. (Alternatively, we could take $x = t+n$ or $y = t+2n$.) $\boxtimes$