Name: Solutions

Directions: Show all work.

1. **[2.5 points]** How many ways are there to make a list $(x_1, \ldots, x_5)$ where each x_i is a digit in $\{0, \ldots, 9\}$ and we have 4 even digits and one odd digit? For example, $(0, 0, 1, 8, 6)$ and $(2, 2, 4, 6, 3)$ count but $(0, 1, 2, 3, 4)$ and $(0, 0, 0, 0, 0)$ do not.

Rule of product.

① Choose position for odd digit $\_\ \_\ \_\ \_\ \_$ (5 options)

② Choose an odd digit from $\{1,3,5,7,9\}$ and put it in the designated place $\_\ 3\ \_\ \_\ \_$ (5 options)

③ Choose a sequence of 4 even digits and put these in the blank positions $2\ 3\ 0\ 0\ 8$ (5^4 options)

5 opts, 5, 5, 5

So the total # is $5 \cdot 5 \cdot 5^4 = \boxed{5^6} = (5^3)^2 = (125)^2 = (100)^2 + 2 \cdot 100 \cdot 25 + 5^4$
$= 15000 + (20+5)^2 = \boxed{15625}$

2. **[2.5 points]** How many integers in $\{1, \ldots, 1000\}$ have no repeated consecutive digits?

We use rule of sum. Let $A = \{n \in \{1, \ldots, 1000\} : n \text{ has no repeated digits}\}$, and for $1 \le i \le 3$, let $A_i = \{n \in A : n \text{ has } i \text{ digits}\}$; note that 1000 has repeated consecutive digits, so we discard it.

Now $|A_1| = |\{1, \ldots, 9\}| = 9$. To count A_2, we use the rule of product:

① choose first digit from $\{1, \ldots, 9\}$ $4\ \_$ (9 options)

② choose second digit from $\{0, \ldots, 9\}$ differently from the digit chosen before $4\ 7$ (9 options)

$\{0, \ldots, 9\} - \{4\}$

So $|A_2| = 9 \cdot 9$. Similarly $|A_3| = 9 \cdot 9 \cdot 9$ since the first digit has 9 options, and the second and 3rd also have 9 options (avoiding the entry selected for the prior position).

So $|A| = |A_1| + |A_2| + |A_3| = 9 + 9^2 + 9^3 = 9(1 + 9 + 9^2) = 9(91) = \boxed{819}$.

3. **[2.5 points]** Count the functions $f\colon \{1,\dots,9\} \to \{1,2,3,4\}$ such that $f(1) = f(t)$ for some $t \in \{2,\dots,9\}$.

Count the complement. Let $\mathcal{U}$ be the set of all functions $f: \{1,\dots,9\} \to \{1,2,3,4\}$,

Let $A = \{f \in \mathcal{U} : f(1) = f(t) \text{ for some } t \in \{2,\dots,9\}\}$. Note that $\bar{A} = \{f \in \mathcal{U} : f(1) \neq f(t) \text{ for all } t \in \{2,\dots,9\}\}$.

We count $\bar{A}$ using the rule of product:

1. Choose value for $f(1)$ in $\{1,2,3,4\}$ (4 options)
2. Choose values for $f(2),\dots,f(9)$ so these are not equal to $f(1)$. (3^8 options)

Also, note that $|\mathcal{U}| = 4^9$ since we choose a sequence of 9 values for $f(1),\dots,f(9)$, each of which is in $\{1,2,3,4\}$ (so 4 options in each of 9 stages).

So $|A| = |\mathcal{U}| - |\bar{A}| = 4^9 - 4\cdot 3^8 = \boxed{4(4^8 - 3^8)}$.

4. **[2.5 points]** How many integers in $\{1000,\dots,5000\}$ are even and have all distinct digits?

$\underset{\{1,\dots,4\}}{\_}\ \underset{\{0,\dots,9\}}{\_}\ \underset{\{0,\dots,9\}}{\_}\ \underset{\{0,2,4,6,8\}}{\_}$

Tricky: the choice for the first digit affects the number of choices for the 4th digit, depending if 2,4 is chosen or 1,3.

Let A be the integers in $\{1000,\dots,4999\}$ that are even and have all distinct digits, let A_1 be the set of $n \in A$ that begin 2 or 4, and let A_2 be the set of $n \in A$ that begin 1,3.

Now for A_1, choose the first digit in $\{2,4\}$, then choose the 4th digit in $\{0,2,4,6,8\}$ differently from the first digit (4 options), then the 2nd digit in $\{0,\dots,9\}$ distinct from 1st and 4th (8 options), and finally the 3rd digit in $\{0,\dots,9\}$ differently from the previously chosen digits (7 options).

Similarly, for A_2: choose 1st digit in $\{1,3\}$ (2 opts)

" 4th digit in $\{0,2,4,6,8\}$ (5 opts)

" 2nd digit differently from 1st and 4th (8 opts)

" 3rd " " " 1st, 4th, 2nd (7 opts).

$$\begin{array}{r} 56 \\ \times 18 \\ \hline 448 \\ 560 \\ \hline 1008 \end{array}$$

So $|A| = |A_1| + |A_2| = 2\cdot4\cdot8\cdot7 + 2\cdot5\cdot8\cdot7 = 2\cdot7\cdot8(4+5) = \boxed{2\cdot7\cdot8\cdot9} = \boxed{1008}$