

Name: Solutions

Directions: Show all work.

1. [2.5 points] How many ways are there to make a list (x_1, \dots, x_5) where each x_i is a digit in $\{0, \dots, 9\}$ and we have 4 even digits and one odd digit? For example, $(0, 0, 1, 8, 6)$ and $(2, 2, 4, 6, 3)$ count but $(0, 1, 2, 3, 4)$ and $(0, 0, 0, 0, 0)$ do not.

Rule of product.

① Choose position for odd digit $_ _ _ _ _$ (5 options)

② Choose an odd digit from $\{1, 3, 5, 7, 9\}$ and put it in the designated place $_ _ _ _ _$ (5 options)

③ Choose a sequence of 4 even digits and put these in the blank positions $_ _ _ _ _$ (5⁴ options)

$$\text{So the total \# is } 5 \cdot 5 \cdot 5^4 = \boxed{5^6} = (5^3)^2 = (125)^2 = (100)^2 + 2 \cdot 100 \cdot 25 + 5^4 \\ = 15000 + (20+5)^2 = \boxed{15625}$$

2. [2.5 points] How many integers in $\{1, \dots, 1000\}$ have no repeated consecutive digits?

We use rule of sum. Let $A = \{n \in \{1, \dots, 1000\} : n \text{ has no repeated digits}\}$, and for $1 \leq i \leq 3$, let $A_i = \{n \in A : n \text{ has } i \text{ digits}\}$; note that 1000 has repeated consecutive digits, so we discard it.

Now $|A_1| = |\{1, \dots, 9\}| = 9$. To count A_2 , we use the rule of product:

① Choose first digit from $\{1, \dots, 9\}$ $_ _$ (9 options)

② Choose second digit from $\{0, \dots, 9\}$ differently from the digit chosen before $_ _$ (9 options)

So $|A_2| = 9 \cdot 9$. Similarly $|A_3| = 9 \cdot 9 \cdot 9$ since the first digit has 9 options, and the second and 3rd also have 9 options (avoiding the entry selected for the prior position).

$$\text{So } |A| = |A_1| + |A_2| + |A_3| = 9 + 9^2 + 9^3 = 9(1 + 9 + 9^2) = 9(91) = \boxed{819}.$$

3. [2.5 points] Count the functions $f: \{1, \dots, 9\} \rightarrow \{1, 2, 3, 4\}$ such that $f(1) = f(t)$ for some $t \in \{2, \dots, 9\}$.

Count the complement. Let U be the set of all functions $f: \{1, \dots, 9\} \rightarrow \{1, 2, 3, 4\}$,

let $A = \{f \in U: f(1) = f(t) \text{ for some } t \in \{2, \dots, 9\}\}$. Note that $\bar{A} = \{f \in U: f(1) \neq f(t) \text{ for all } t \in \{2, \dots, 9\}\}$.

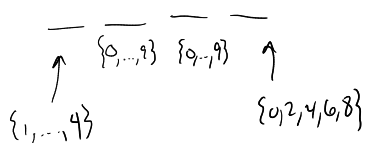
We count \bar{A} using the rule of products:

- ① choose value for $f(1)$ in $\{1, 2, 3, 4\}$ (4 options)
- ② choose values for $f(2), \dots, f(9)$ (so these are not equal to $f(1)$). (3^8 options)

Also, note that $|U| = 4^9$ since we choose a sequence of 9 values for $f(1), \dots, f(9)$, each of which is in $\{1, 2, 3, 4\}$ (so 4 options in each of 9 stages).

$$\text{So } |A| = |U| - |\bar{A}| = 4^9 - 4 \cdot 3^8 = \boxed{4(4^8 - 3^8)}.$$

4. [2.5 points] How many integers in $\{1000, \dots, 5000\}$ are even and have all distinct digits?



Tricky: the choice for the first digit affects the number of choices for the 4th digit, depending if 2, 4 is chosen or 1, 3.

Let A be the integers in $\{1000, \dots, 4999\}$ that are even and have all distinct digits, let A_1 be the set of $n \in A$ that begin 2 or 4, and let A_2 be the set of $n \in A$ that begin 1, 3.

Now for A_1 , choose the first digit in $\{2, 4\}$, then choose the 4th digit in $\{0, 2, 4, 6, 8\}$ differently from the first digit (4 options), then the 2nd digit in $\{0, \dots, 9\}$ distinct from 1st and 4th (8 options), and finally the 3rd digit in $\{0, \dots, 9\}$ differently from the previously chosen digits (7 options).

Similarly, for A_2 : choose 1st digit in $\{1, 3\}$ (2 opts)
 " 4th digit in $\{0, 2, 4, 6, 8\}$ (5 opts)
 " 2nd digit differently from 1st and 4th (8 opts)
 " 3rd " " " 1st, 4th, 2nd (7 opts).

$$\begin{array}{r} 4 \\ 56 \\ 18 \\ \hline 448 \\ 560 \\ \hline 1008 \end{array}$$

$$\text{So } |A| = |A_1| + |A_2| = 2 \cdot 4 \cdot 8 \cdot 7 + 2 \cdot 5 \cdot 8 \cdot 7 = 2 \cdot 7 \cdot 8(4+5) = \boxed{2 \cdot 7 \cdot 8 \cdot 9} = \boxed{1008}$$