Name: _Solutions_

Directions: Show all work.

1. [**2 points**] Give a list of distinct integers of maximum possible length that does not contain an increasing subsequence of length 5 or a decreasing subsequence of length 4.

By Erdős–Szekeres, a sequence of length $4 \cdot 3 + 1$ contains an increasing subsequence of length $4+1$ or a decreasing subsequence of length $3+1$. Hence the maximum possible length is at most $4 \cdot 3$, or 12.

Let L be $\boxed{9, 10, 11, 12, \ 5,6,7,8, \ 1,2,3,4}$ as let $X_1 = 9, 10, 11, 12$, $X_2 = 5, 6, 7, 8$, and $X_3 = 1, 2, 3, 4$. Since elements in distinct X_i are decreasing, an increasing subsequence is contained in one of the parts and has size at most 4. A decreasing subsequence has at most one element in each part and hence has size at most 3.

2. [**2 points**] Give a formula for the number of edges in the n-vertex complete graph K_n. Prove that your answer is correct.
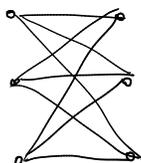
Since each vertex in K_n has degree $n-1$, we have

$$\sum_{v \in V(K_n)} d(v) = \sum_{v \in V(K_n)} n-1 = |V(K_n)| \cdot (n-1) = n(n-1).$$

By the degree-sum formula, $\sum_{v \in V(K_n)} d(v) = 2|E(K_n)|$. Therefore $2|E(K_n)| = n(n-1)$
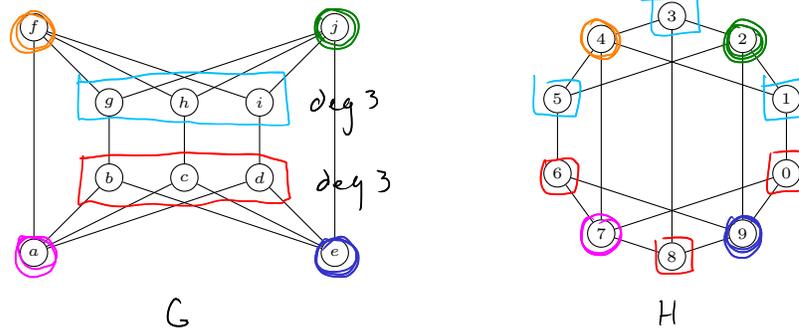
and so $\boxed{|E(K_n)| = \dfrac{n(n-1)}{2}}$.

3. [**2 points**] Construct a 6-vertex bipartite graph with 9 edges.

$K_{3,3}$

4. [**2 points**] Are the following graphs isomorphic? If so, then give an isomorphism. If not, then give a property that distinguishes the graphs.

G H

Yes, these graphs are isomorphic, for example via the isomorphism

V(G)	a	b	c	d	e	f	g	h	i	j
V(H)	7	6	8	0	9	4	5	3	1	2

There are 24 isomorphisms in total, do you see why?

5. [**2 points**] Let G be an n-vertex m-edge graph. Prove that G contains a vertex u whose degree $d(u)$ is at least $2m/n$.

By the degree sum formula, we have $\sum_{v \in V(G)} d(v) = 2m$, and so $\frac{1}{n} \sum_{v \in V(G)} d(v) = \frac{2m}{n}$.

Therefore the average degree in G is $\frac{2m}{n}$, and at least one of the vertices in G has degree at least the average.