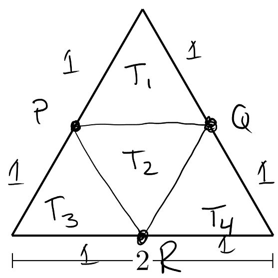
Name: Solutions

Directions: Show all work.

1. **[4 points]** Let S be a set of 5 points in an equalaterial triangle with side length 2. Prove that S contains a pair of points that are at distance at most 1.

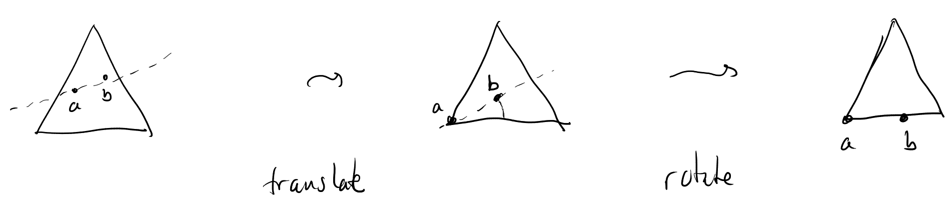


Note: underlined parts are not required for full credit but are nonetheless included here for illustration.

Let T be the equilateral triangle with side length 2. Let P, Q, and R be the midpoints of the edges of T. Drawing the segments PQ, QR, and RP partitions T into four triangles T_1, T_2, T_3, T_4. Since each T_i has two edges of unit length that meet at the same angle as in T, it follows from Side-Angle-Side similarity that each T_i is similar to T and hence each T_i is equilateral with side length 1.

By the pigeonhole principle, since $|S|=5$ and $\{T_1, \dots, T_4\}$ is a partition of size 4, there exist distinct points $p, q \in S$ that belong to the same part T_j.

We show that every pair of points in T_j is at distance at most 1. If $a, b \in T_j$, then we may translate $\{a,b\}$ until one of those points is at a vertex, and then rotate



until both a, b are on an edge of T_j. Since each edge has length 1, $\mathrm{dist}(a,b) \leq 1$. It follows that $\mathrm{dist}(p,q) \leq 1$ as required.

2. **[2 parts, 3 points each]** Recall that two integers x and y are *relatively prime* if their greatest common divisor is 1.

   (a) Find a maximum size subset of $\{1, \ldots, 8\}$ such that no two elements in A are relatively prime. Prove your answer is correct.

Let $A = \{2,4,6,8\}$. Note that A is a subset of size 4 and since each number in this set is divisible by 2, no pair of elements in A is relatively prime. Hence the maximum size is at least 4. If A' is a subset with $|A'| \geq 5$, then A' contains an odd number. But every odd number in $\{1, \ldots, 8\}$ is relatively prime to all other integers in $\{1, \ldots, 8\}$, so no such subset with more than 4 elements exists.

   (b) Find a maximum size subset of $\{1, \ldots, 2n\}$ such that no two elements in A are relatively prime. Prove your answer is correct.

Let $A = \{2, 4, \ldots, 2n\}$. As in (a), every elt in A is divisible by 2 and so no pair is relatively prime. Note that $|A| = n$.

It remains to show that if $|B| \geq n+1$ and $B \subseteq \{1, \ldots, 2n\}$, then B contains a pair of relatively prime integers. Note that for all integers k, we have $\gcd(k, k+1) = 1$. This is because every divisor of both $k+1$ and k would have to divide their difference, and $(k+1) - k = 1$.

Partition $\{1, \ldots, 2n\}$ into the parts $\{1,2\}, \{3,4\}, \ldots, \{2n-1, 2n\}$. There are n parts. Since $|B| \geq n+1$, by the pigeonhole principle, it must be that B contains both elements of same part, and so $k, k+1 \in B$ for some integer k. As we have seen, $\gcd(k, k+1) = 1$ and so $k, k+1$ is a pair of relatively prime elements in B.