Name: Solutions

Directions: Show all work.

1. In a game of Nim, tokens are placed in various piles. In each turn, players removing one or more tokens from a selected pile. In this variant, the game begins with n piles, each starting with 10 tokens. When no tokens remain, the player to move loses.

   (a) **[3 points]** Prove that if n is even, then Player 2 has a winning strategy. (Hint: Analyzing the general game of Nim is tricky; give an explicit strategy for Player 2 in this special case.)

Player 2 divides the piles into $\frac{n}{2}$ pairs: $\{A_1, B_1\}, \ldots, \{A_{n/2}, B_{n/2}\}$.

Player 2 maintains the invariant that, at the start of Player 1's turn, for each i, the number of tokens in A_i and B_i are equal. If Player 1 removes t tokens from A_i or B_i, then Player 2 responds by removing t tokens from the other pile in $\{A_i, B_i\}$. This restores the invariant. Since the number of tokens in A_i and B_i are equal, if it is legal for Player 1 to remove t tokens from one of these piles, then Player 2 may remove t tokens from the other pile. Since Player 2 always has a response, this is a winning strategy for Player 2.

   (b) **[2 points]** Use part (a) to show that if n is odd, then Player 1 has a winning strategy.

If n is odd, then Player 1 first removes all 10 tokens from 1 pile, leaving Player 2 as next to move in a state with an even number (namely, $n-1$) of piles with 10 tokens in each pile. Now Player 1 can use the winning strategy from part (a) to force a win.

2. **[5 points]** Let $a_0 = 0$ and $a_n = \frac{1}{2-a_{n-1}}$ for $n \geq 1$. Guess a formula for a_n and prove your formula is correct.

$a_1 = \frac{1}{2-0} = \frac{1}{2}$

$a_2 = \frac{1}{2-\frac{1}{2}} = \frac{2}{4-1} = \frac{2}{3}$

$a_3 = \frac{1}{2-\frac{2}{3}} = \frac{3}{6-2} = \frac{3}{4}$

Claim: $\boxed{a_n = \frac{n}{n+1}}$ for $n \geq 0$.

Pf: By induction on n. Basis Step: If $n=0$, then $a_0 = 0$ and $\frac{0}{0+1} = 0$, and so the formula holds when $n=0$.

Inductive Step: Suppose $n \geq 1$. By the inductive hypothesis, we have $a_{n-1} = \frac{n-1}{n}$.

Therefore $a_n = \frac{1}{2-a_{n-1}} = \frac{1}{2-\frac{n-1}{n}} = \frac{n}{2n-(n-1)} = \frac{n}{n+1}$. ☒