Name: Solutions

Directions: Show all work. Unless otherwise directed, you may leave your answer in terms of factorials, falling factorials, and binomial/multinomial coefficients. Any sums or products should be as simple as possible.

1. **[2.5 points]** How many 3-element subsets of $\underbrace{\{a, b, c, d, e, f, g, h\}}_{\text{size } 8}$ are there? Give an explicit numerical answer.

$$\binom{8}{3} = \frac{8!}{3! \cdot 5!} = \frac{8_{(3)}}{3!} = \frac{8 \cdot 7 \cdot \cancel{6}}{\cancel{3 \cdot 2 \cdot 1}} = \boxed{56}$$

2. **[2.5 points]** A group of 50 people must be split into two teams of size 5 and five teams of size 8. How many ways can this be done?

With ordered teams: $\binom{50}{5,5,8,8,8,8,8} = \frac{50!}{(5!)^2(8!)^5}$

Unorder the teams: $\underbrace{\frac{1}{2!}}_{\text{two teams of size 5}} \cdot \underbrace{\frac{1}{5!}}_{\text{five teams of size 8}} \cdot \frac{50!}{(5!)^2(8!)^5} = \boxed{\frac{50!}{(5!)^3(8!)^5 \cdot 2!}}$

3. **[2.5 points]** A random lattice path of length $2n$ is generated by starting from $(0, 0)$. At each step, we increase the x-component by 1 with probability $1/2$ and we increase the y-component by 1 with probability $1/2$. What is the probability the lattice path ends at (n, n)?

Let $\Omega = \{x,y\}^{2n} = \{(s_1, \dots, s_{2n}) : \text{each } s_i \in \{x,y\}\}$. Note that $|\Omega| = 2^{(2n)} = 4^n$.

Let A be the event that the lattice path ends at (n,n). Since there are $\binom{2n}{n}$ lattice paths that end in (n,n), we have $Pr(A) = \frac{|A|}{|\Omega|} = \boxed{\frac{\binom{2n}{n}}{4^n}}$.

4. **[2.5 points]** A standard deck of cards has one card for each of the suit/rank pairs. The suits are spades, hearts, diamonds, and clubs; the ranks are ace, 2 through 10, jack, queen, and king. What is the probability that a poker hand (i.e. a set of 5 cards) has at most 2 cards in each suit?

Soln 1: Rule of sum

Case 1: 2 1 1 1

- One suit with 2 cards $[4\binom{13}{2}$ options$]$
- other 3 suits each have 1 card $[13^3$ options$]$

Case 2: 2 2 1 _

- two suits with 2 cards $[\binom{4}{2}\binom{13}{2}^2$ options$]$
- one more card from remaining two suits [26 opts]

So Prob $= \boxed{\frac{4\binom{13}{2} \cdot 13^3 + \binom{4}{2}\binom{13}{2}^2 \cdot 26}{\binom{52}{5}}} = \frac{5239}{8330} \approx 0.6289$

Soln 2: Count complement: at least 3 cards in one suit. Note: only one suit can have ≥ 3 cards.

$$\underbrace{4}_{\text{choose majority suit}} \cdot \left[\underbrace{\binom{13}{3}}_{3 \text{ in maj}}\underbrace{\binom{39}{2}}_{2 \text{ others}} + \underbrace{\binom{13}{4}}_{4 \text{ in maj}}\binom{39}{1} + \underbrace{\binom{13}{5}}_{5 \text{ in maj}}\cancel{\binom{39}{0}}^{1} \right]$$

So Prob $= \boxed{\frac{\binom{52}{5} - 4\left(\binom{13}{3}\binom{39}{2} + \binom{13}{4}\binom{39}{1} + \binom{13}{5}\right)}{\binom{52}{5}}} = \frac{5239}{8330} \approx 0.6289$