Name: Solutions

Directions: Show all work.

1. **[6 points]** For $n \geq 0$, let

$$S_n = \sum_{k=1}^{2n} (-1)^k k^2 = -1^2 + 2^2 - 3^2 + 4^2 - \cdots - (2n-1)^2 + (2n)^2.$$

Prove that for each non-negative integer n, we have that $S_n = (2n+1)n$.

Pf: By induction on n. Basis step: If $n=0$, then $S_n = 0$ (since S_n is the empty sum) and $(2n+1)n = (2\cdot 0+1)\cdot 0 = 0$. So the claim holds when $n=0$.

Inductive Step: Suppose $n \geq 1$. By the inductive hypothesis, $S_{n-1} = (2(n-1)+1)(n-1) = (2n-1)(n-1)$. Adding $-(2n-1)^2 + (2n)^2$ to both sides gives

$$\begin{aligned} S_{n-1} - (2n-1)^2 + (2n)^2 &= (2n-1)(n-1) - (2n-1)^2 + (2n)^2 \\ &= (2n-1)\left[(n-1) - (2n-1)\right] + (2n)^2 \\ &= (2n-1)[-n] + (2n)^2 = 4n^2 - n(2n-1) \\ &= n(4n - (2n-1)) = n(2n+1). \end{aligned}$$

Also, $S_{n-1} - (2n-1)^2 + (2n)^2 = \left(-1^2 + 2^2 - \cdots - (2n-3)^2 + (2n-2)^2\right) - (2n-1)^2 + (2n)^2$

$= S_n.$

It follows that $S_n = S_{n-1} - (2n-1)^2 + (2n)^2 = n(2n+1)$. ∎

2. **[2 parts, 2 points each]** Recall that the adjusted Fibonacci sequence is defined by $\hat{F}_0 = \hat{F}_1 = 1$ and $\hat{F}_n = \hat{F}_{n-1} + \hat{F}_{n-2}$ for $n \geq 2$.

   (a) Let n be a positive integer and let t be the maximum integer such that $\hat{F}_t \leq n$. Prove that $n - \hat{F}_t < \hat{F}_{t-1}$. (Hint: there is a short proof, without induction/no minimum counter-example. If stuck, then try a proof by contradiction.)

Pf. First, note that since $n \geq 1 = \hat{F}_1$, we have that $t \geq 1$.

Since t is the maximum integer such that $\hat{F}_t \leq n$, we have that $\hat{F}_{t+1} > n$.

Since $t \geq 1$, we have that $\hat{F}_{t+1} = \hat{F}_t + \hat{F}_{t-1}$ and so $\hat{F}_t + \hat{F}_{t-1} > n$.

Rearranging gives $n - \hat{F}_t < \hat{F}_{t-1}$.

   (b) Prove that each positive integer n is a sum of distinct, non-consecutive adjusted Fibonacci numbers. For example, if $n = 20$, then we have $20 = 13 + 5 + 2 = \hat{F}_6 + \hat{F}_4 + \hat{F}_2$.

Pf. By induction on n. Basis Step: If $n=1$, then $1 = \hat{F}_1$, and so n is the sum of distinct, non-concecutive Fibonacci numbers.

Inductive Step: Suppose $n \geq 2$. Let t be the maximum integer such that $\hat{F}_t \leq n$. Since $n \geq 2 = \hat{F}_2$, we know that $t \geq 2$ and $\hat{F}_2 \geq 2$. If n is a Fibonacci number, then the claim is satisfied by the sum with just the single term n. Otherwise $\hat{F}_t < n$ and so $0 < n - \hat{F}_t \leq n-2$. Let $m = n - \hat{F}_t$. Since $1 \leq m < n$, the inductive hypothesis implies that m is a sum S of distinct non-consecutive Fibonacci numbers; since $m < \hat{F}_{t-1}$ by part (a), all terms in S are at most $\hat{F}_{t-2}$. By adding $\hat{F}_t$ to the sum S, we obtain n as a sum of distinct non-consecutive Fibonacci numbers. ☒