Directions: Solve the following problems. All written work must be your own. See the course syllabus for detailed rules.

1. [2.1.6] You have a 3×3-square, and you throw 10 darts at it. Show that no matter where the darts land, there are two darts whose distance is at most $\sqrt{2}$.

2. [2.1.22] I have 51 rectangular pieces of cardboard, each of which has an integer length and width in the set $\{1, \ldots, 100\}$. (Note that squares are allowed.) Prove that there are two rectangles such that one can fully cover the other when placed on top.

3. Let $a_0 = 4$, $a_1 = 10$, $a_2 = 88$, and $a_n = a_{n-1} + 21a_{n-2} - 45a_{n-3}$ for $n \geq 3$. Use the characteristic equation method to solve the recurrence.

4. For positive m and n, a domino tiling of an $m \times n$ grid is *rigid* if every horizontal and vertical cut crosses a domino. In this problem, we characterize the grids with even dimensions that have rigid domino tilings.

   (a) Prove that if $m = 2r$, $n = 2s$, and the $m \times n$ grid has a rigid domino tiling, then $(r - 2)(s - 2) \geq 2$. (Hint: generalize our argument in class that the 6×6 grid has no rigid domino tiling.)

   (b) Construct a rigid domino tiling of the 6×8 grid.

   (c) Prove that if $m \geq 3$ and the $m \times n$ grid has a rigid domino tiling, then the $m \times (n + 2)$ grid also has a rigid domino tiling. (Hint: show how to modify an arbitrary rigid domino tiling of the $m \times n$ grid to obtain a rigid domino tiling of the $m \times (n + 2)$ grid.)

   (d) Let m and n be positive, even integers with $m \leq n$. Prove that the $m \times n$ grid has a rigid domino tiling if and only if $m \geq 6$ and $n \geq 8$. (Hint: for the forward direction, use part (a) to give a direct proof. For the backward direction, use parts (b) and (c) to give an inductive proof.)