Directions: Solve the following problems. All written work must be your own. See the course syllabus for detailed rules.

1. Let a_n be the number of lists of length n with entries in $\{0, 1, 2\}$ without two consecutive zeros. Note that $a_0 = 1$ (since the empty list does not have consecutive zeros), $a_1 = 3$, and $a_2 = 8$ (since all 9 lists of length 2 are counted except 00).

   (a) Find a second order homogeneous recurrence relation for a_n. In other words, find constants s and t such that $a_n = sa_{n-1} + ta_{n-2}$ for $n \geq 2$. Remember to include base cases and argue that your recurrence relation is correct.

   (b) Use part (a) to explicitly compute a_n for $0 \leq n \leq 6$.

   (c) Use the characteristic equation method to solve your recurrence in part (a) to find an explicit formula for a_n.

2. Prove that if it is possible to tile an $m \times n$ grid with 4×1 rectangular tiles, then at least one of the side lengths is divisible by 4. (Hint: find a way to color the grid with 4 colors so that each tile covers one cell of each color.)

3. For $b \geq 0$, let b_n be the number of ways to tile a $3 \times n$ grid with 1×3 rectangular tiles. Note that $b_0 = 1$, since placing zero tiles counts as a tiling of the 3×0 grid.

   (a) Find a recurrence relation for b_n. (Your recurrence should include all needed base cases.)

   (b) Recall that the number of ways a_n of tiling a $2 \times n$ grid with dominos is given by the recurrence $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. How does b_n compare with a_n, the number of ways to tile a $2 \times n$ grid with dominos? Explain. Can you prove your claim?

4. [SS 1.3.{8,9}] You work at a car dealership that sells three models: A pickup trick, an SUV, and a compact hybrid. Your job is to park the vehicles in a row. The pickup trucks and the SUVs take up two spaces while the hybrid takes up one space. Let n be a nonnegative integer and let $f(n)$ be the number of ways of arranging vehicles in exactly n spaces, with each space occupied.

   (a) Find a recurrence relation for $f(n)$ and use it to compute $f(0)$ through $f(10)$.

   (b) Find a first-order recurrence relation g that appears to match f (i.e. $g(n)$ should depend only on $g(n-1)$).

   (c) Prove that $g(n) = f(n)$ by induction.

   (d) Use the values for $f(0)$ and $f(1)$ to find a candidate formula for $f(n)$ of the form $f(n) = a2^n + b(-1)^n$. Prove that your formula is correct.