

**Directions:** Solve the following problems. All written work must be your own. See the course syllabus for detailed rules.

1. [5.1.5] Let  $c \leq b \leq a$  be non-negative integers. Give two proofs, one algebraic and the other combinatorial, for  $\binom{a}{b} \binom{b}{c} = \binom{a}{c} \binom{a-c}{b-c}$ .
2. Recall that Pascal's identity for binomial coefficients is  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$  and tells the story that all  $k$ -element subsets of a set of size  $n$  either include or omit the last element.  
Suppose  $a, b, c$  are non-negative integers summing to  $n$ . What equation tells the story that in a partition of  $[n]$  into 3 labeled parts of sizes  $a, b$ , and  $c$ , the last element  $n$  belongs to one of the three parts? Explain.
3. Give a combinatorial proof of the identity  $\sum_{t=k}^n \binom{t}{k} = \binom{n+1}{k+1}$ . (Hint: let  $A$  be the set of all  $(k+1)$ -element subsets of  $[n+1]$ . Group the sets in  $A$  by their maximum element. Look at, for example,  $n=5$  and  $k=2$  for insight into the general case.)
4. [5.1.22] We have a group of math majors consisting of  $n$  sophomores and  $n$  juniors. We want to form a smaller group that has a total of  $n$  students in it, but from among that group we want to designate one of the students to serve as a departmental liaison. The liaison needs to be a junior, but there is no other restriction on the students chosen for the smaller group.
  - (a) In how many different ways can we form the smaller group with a junior liaison?
  - (b) Let  $k$  be a positive integer. In how many ways can we pick  $k$  juniors, a liaison from among the  $k$  juniors, and  $n-k$  sophomores?
  - (c) Use parts (a) and (b) to give a simple expression for  $\sum_{k=0}^n k \binom{n}{k}^2$ .
5. [5.1.16] Using algebra, find and prove an identity of the form  $\sum_{k=0}^n \frac{(2n)!}{k!^2 (n-k)!^2} = \binom{?}{n}^2$ . (Hint: in the terms on the LHS, multiply the numerator and denominator by  $n!^2$ .)
6. [5.1.30] Recall that  $k! \geq \left(\frac{k}{e}\right)^k$ . Use this to prove that for  $1 \leq k \leq n$ , we have  $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$ .