Name: _____

Directions: Show all work. No credit for answers without work. Except when otherwise directed, you may leave answers in terms of binomial/multinomial coefficients, factorials, and sums with a small number of terms.

1. **[7 points]** How many subsets of $\{1, \ldots, n\}$ have size 4? Give a simple formula in terms of n without using binomial/multinomial coefficients, factorials, or sums.

2. **[7 points]** For $n \geq 2$, what are the odds that a random permutation of $\{1, \ldots, n\}$ begins with two consecutive integers?

3. **[2 parts, 6 points each]** A standard deck of cards has one card for each suit/rank pair, where the suits are spades, hearts, diamonds, and clubs, and the ranks are ace, 2 through 10, jack, queen, and king.

   (a) How many ways are there to choose a set of 6 cards with no spades?

   (b) How many ways are there to choose a set of 6 cards which has at least one card in each of the four suits?

4. **[3 parts, 5 points each]** How many ways are there to arrange the letters in the word KNICKKNACK:

   (a) without any restrictions?

   (b) that does not start or end with a 'K'?

   (c) where the substring 'CKC' appears?

5. **[3 parts, 5 points each]** A donut shop sells 5 different kinds of donuts.

   (a) How many ways are there to purchase 24 donuts? (The order the donuts are purchased is unimportant.)

   (b) How many ways are there to purchase 24 donuts if we must have at least 3 of each type?

   (c) How many ways are there to purchase 24 donuts if we must purchase an even number of each type? (Note that zero is even, so not every type need be purchased.)

6. **[2 parts, 10 points each]** Give combinatorial proofs for each of the following.

   (a) $\binom{n}{k}\binom{k}{2} = \binom{n}{2}\binom{n-2}{k-2}$

   (b) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n.$

7. **[6 points]** How many lattice paths from $(0,0)$ to $(3n,2n)$ avoid the point $(2n,n)$? For example, when $n=1$, the answer is 4.

8. **[2 parts, 6 points each]** A permutation σ of $\{1, \ldots, n\}$ is chosen at random.

   (a) What is the expected value of the first entry in σ?

   (b) What is the expected value of the sum of the first k entries in σ? (Hint: for $1 \leq i \leq n$, let $X_i = i$ if i appears in the first k positions of σ; otherwise let $X_i = 0$.)

9. **[6 points]** How many ways are there to form 2 groups of size 4 and 3 groups of size 5 from a class of 23 students?