Name: Solutions

Directions: Show all work. No credit for answers without work.

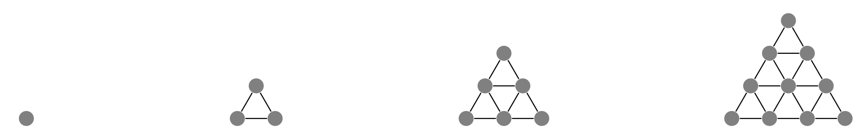
1. **[15 points]** Let $T_1, \ldots, T_n$ be a list of domino tilings of a (2×8)-grid. (Note that each entry in the list is a complete tiling, so for example T_1 might be the tiling that places all n dominos vertically.) What is the minimum n such that two tilings on the list must be identical?

Recall: # tilings of a $(2\times k)$-grid is F_k, where $F_0 = F_1 = 1$, $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$.

k	0	1	2	3	4	5	6	7	8
F_k	1	1	2	3	5	8	13	21	34

Since there are 34 domino tilings of a (2×8)-grid, we need $n \geq \boxed{35}$ to be sure two tilings in the list are the same via the pigeonhole principle.

2. **[2 parts, 5 points each]** The *triangular lattice* G_n is the graph whose vertices are arranged in rows of sizes $1, 2, \ldots, n$, with the midpoints of the rows centered on a common vertical line. Consecutive vertices in the same row are adjacent, and the jth vertex in row i is adjacent to the jth and $(j+1)$st vertex in row $i+1$. No other pairs of vertices are adjacent. See below.



G_1 G_2 G_3 G_4

Note that G_n has $\binom{n+1}{2}$ vertices.

(a) Find a formula for the number of edges in G_n.

Many ways to solve.

Soln 1: By symmetry, there are the same # of edges with each of the 3 slopes. In the j^{th} row, there are j vertices and $j-1$ edges. So

horizontal edges $= 0 + 1 + 2 + \cdots + n-1$
$= \binom{n}{2}$

and so $|E(G_n)| = \boxed{3\binom{n}{2}}$.

Soln 2: Suppose $n \geq 2$: We count degrees.

There are:

$\Rightarrow$ 3 verts of deg 2

$\Rightarrow$ $3(n-2)$ verts of deg 4

$\Rightarrow$ The remaining $\binom{n+1}{2} - 3(n-2) - 3$ verts have deg 6.

So $|E(G)| = \frac{1}{2}\sum_v d(v)$

$= \frac{1}{2}\Big(3\cdot 2 + 3(n-2)\cdot 4 + \Big[\binom{n+1}{2} - 3(n-2) - 3\Big]\cdot 6\Big)$

$= 3 + 6(n-2) + 3\binom{n+1}{2} - 9 - 9(n-2)$

$= 3\binom{n+1}{2} - 3(n-2) - 6$

$= 3\left[\binom{n+1}{2} - (n-2) - 2\right]$

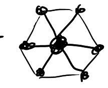
$= 3\left[\frac{(n+1)n}{2} - n\right] = 3\binom{n}{2}$

(b) Let d_n be the average of the degrees of vertices in G_n. Find a formula for d_n. What is $\lim_{n\to\infty} d_n$? Does this make sense?

We have $d_n = \frac{1}{|V(G)|}\sum_v d(v) = \frac{1}{\binom{n+1}{2}}\cdot 2|E(G)| = \frac{1}{\binom{n+1}{2}}\cdot 2\left[3\binom{n}{2}\right] = \frac{6}{\frac{(n+1)(n)}{2}}\cdot\frac{n(n-1)}{2} = \frac{6\cdot \cancel{2}}{(n+1)\cancel{(n)}}\cdot\frac{\cancel{n}(n-1)}{\cancel{2}}$

$= 6\cdot\frac{n-1}{n+1} = \boxed{6\left(1 - \frac{2}{n+1}\right)}$.

So $\boxed{\lim_{n\to\infty} d_n = 6}$. This makes sense.

For large n, almost all vertices of G_n are in the interior  and have degree 6.

3. Let n be a positive integer and suppose that $A \subseteq \{1, 2, \ldots, 5n\}$.

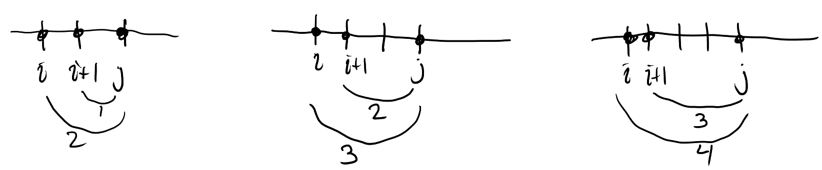
(a) **[15 points]** Show that if $|A| > 2n$, then there exists $x, y \in A$ such that $y - x = 2$ or $y - x = 3$. (Hint: partition $\{1, \ldots, 5n\}$ into n intervals, each of size 5.)

Pf. Partition $\{1, 2, \ldots, 5n\}$ into $\{X_1, \ldots, X_n\}$ where $X_j = \{5(j-1)+1, \ldots, 5(j-1)+5\}$.

Since $|A| > 2n$, it follows from the pigeonhole principle that $|A \cap X_j| > 2$ for some j. This means that there is an interval I of 5 consecutive integers in $\{1, \ldots, 5n\}$ in which A has at least 3 elements. Suppose $I = \{z+1, z+2, \ldots, z+5\}$ and $|A \cap I| \geq 3$.

Case 1: A has no consecutive elements in I. We have $A \cap I = \{z+1, z+3, z+5\}$ and so A contains $z+3$ and $z+1$ whose difference is 2.

Case 2: A has consecutive elements $i, i+1$ in I as well as another element j. Note that $\{|j-i|, |j-(i+1)|\}$ is either $\{1,2\}$, $\{2,3\}$, or $\{3,4\}$. In all cases, A has a pair of elements with difference 2 or 3. ▨



(b) **[10 points]** Show that if $|A| = 2n$, then the conclusion in part (a) need not hold.

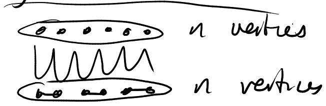
Let A be the set of all integers in $\{1, \ldots, 5n\}$ of the form $5k+1$ or $5k+2$ where $0 \leq k \leq n-1$, so that $A = \{1, 2, 6, 7, 11, 12, 16, 17, \ldots, 5(n-1)+1, 5(n-1)+2\}$.

Note that $|A| = 2n$ and for distinct $x, y \in A$, we have that $|y-x|$ is either 1 (if x and y are consecutive) or $|y-x| \geq 4$ otherwise. Hence there does not exist $x, y \in A$ with $y - x = 2$ or $y - x = 3$. ▨

4. **[10 points]** Let n be a positive integer. Prove that there exists a $2n$-vertex graph with n vertices of degree n and n vertices of degree $n+1$ if and only if n is even.

($\Rightarrow$) Every graph has an even number of vertices of odd degree, by the Handshaking Lemma. Since one of $\{n, n+1\}$ is even and the other is odd, it follows that such a graph has n vertices of odd degree and n vertices of even degree. Therefore n must be even.

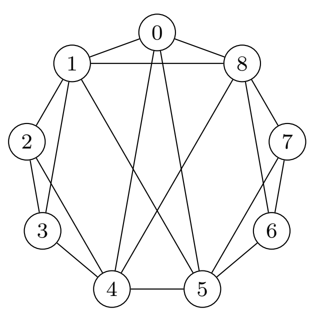
($\Leftarrow$) If n is even, then we may add a perfect matching to one part of $K_{n,n}$ to obtain a $2n$-vertex graph with n vertices of degree n and n vertices of degree $n+1$.



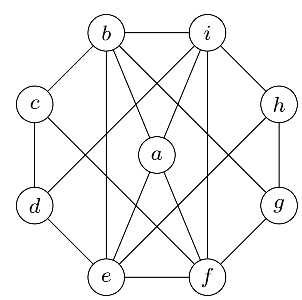
5. **[5 points]** Give the definition of a *bipartite graph*.

A graph G is bipartite if $V(G)$ can be partitioned into parts X and Y such that each edge in G has one endpoint in X and the other endpoint in Y.

6. **[10 points]** Are the following graphs isomorphic? Either give an isomorphism or explain why not.



G

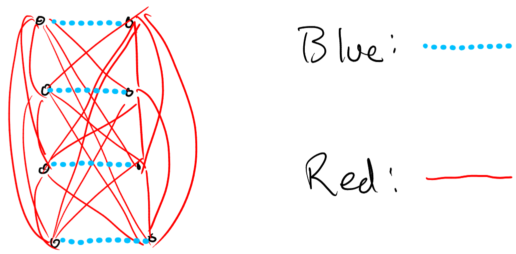


H

$\boxed{\text{No.}}$ Note that 0 is the only vertex in G of degree 4 and a is the only vertex in H of degree 4. Since $H-a$ is bipartite but $G-0$ is not bipartite (it contains the triangle 123, for example), these two graphs cannot be isomorphic.

7. **[25 points]** Recall that P_3 is the path on 3 vertices. Show that $r(P_3, K_5) = 9$. Be sure to show both that $r(P_3, K_5) > 8$ and $r(P_3, K_5) \leq 9$.

Pf. First, we show $r(P_3, K_5) > 8$ by giving a {blue, red}-edge-coloring G of K_8 that avoids a blue P_3 and avoids a red K_5. The blue subgraph of G consists of 4 edges with distinct endpoints; all other edges are red:



Clearly there is no blue P_3. Also, every red complete subgraph must omit one endpoint from each of the 4 blue edges, and so the red complete subgraphs have at most 4 vertices. Hence G has no red K_5, and $K_8 \not\to (P_3, K_5)$.

Next, we prove $r(P_3, K_5) \leq 9$. Let G be a {blue, red}-edge-coloring of K_9. If a vertex in G has blue degree ≥ 2, then G has a blue P_3 and we are done. Otherwise the number of blue edges is at most $\frac{1}{2}\sum_v d_b(v) \leq \frac{1}{2}\cdot 9 = 4.5$.

Hence G has at most 4 blue edges. Deleting an endpoint from each blue edge leaves a red complete subgraph on at least 5 vertices, and so G contains a red K_5. ∎