Math378 Test 1 **Fri. Feb. 5, 2025**

Name: Solutions

Directions: Show all work. No credit for answers without work.

1. Prove the following by induction or the no minimum counter-example technique.

   (a) **[15 points]** If $n \geq 0$, then $\sum_{k=1}^{n}(k^2+2k-1)2^{k-1} = n^2 2^n$.

By induction on n.

Basis Step: If $n=0$, then the LHS is 0 (empty sum) and the RHS is $0^2 \cdot 2^0 = 0$. So the identity holds.

Inductive Step: Suppose $n \geq 1$. By the inductive hypothesis, $\sum_{k=1}^{n-1}(k^2+2k-1)2^{k-1}$ equals $(n-1)^2 2^{n-1}$. Adding $(n^2+2n-1)2^{n-1}$ to both sides gives

$$\begin{aligned}\sum_{k=1}^{n}(k^2+2k-1)2^{k-1} &= (n-1)^2 2^{n-1} + (n^2+2n-1)2^{n-1} \\ &= \left[(n^2-2n+1)+(n^2+2n-1)\right]2^{n-1} \\ &= 2n^2 2^{n-1} = n^2 2^n\end{aligned}$$

$\square$

$1 \leq 2^{n-1}$

scratch:

want $n \leq 2n-2$

$2 \leq n$.

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   (b) **[15 points]** If $n \geq 0$, then $n \leq 2^n$.

By induction on n. If $n=0$, then the LHS is 0 and the RHS is 2^0 or 1, and so the inequality holds.

Inductive Step: Suppose $n \geq 1$. By the inductive hypothesis, we have $n-1 < 2^{n-1}$, or $n < 2^{n-1}+1$. Since 2^{n-1} is the product of $n-1$ copies of 2 and $n-1 \geq 0$, it follows that $2^{n-1} \geq 1$. Therefore

$$n < 2^{n-1}+1 \leq 2^{n-1}+2^{n-1} = 2 \cdot 2^{n-1} = 2^n$$

and so the inequality follows.

$190 = (4)(20) + (10)(11)$

2. **[20 points]** A store sells 11-cent and 20-cent stamps. Prove that for each integer n such that $n \geq 190$, it is possible to purchase some combination of stamps with total postage n cents. (Hints: consider a proof by induction on n with basis step $n = 190$. It may be helpful to note that $(5)(20) + (-9)(11) = 1$ and $(-6)(20) + (11)(11) = 1$.)

Pf. By induction on n.

Basis Step: If $n = 190$, then we obtain n cents of postage by purchasing 4 20-cent stamps and 10 11-cent stamps, since

$$(4)(20) + (10)(11) = 80 + 110 = 190.$$

Inductive Step: Suppose that $n \geq 191$. By the inductive hypothesis, it is possible to purchase postage worth $n-1$ cents, and so

$$20r + 11s = n-1 \qquad (*)$$

for some nonnegative integers r and s. Note that $r \leq 5$ and $s \leq 8$ is impossible, since then $190 \leq n-1 = 20r + 11s \leq (20)(5) + (11)(8) = 188$, a contradiction. Therefore $r \geq 6$ or $s \geq 9$.

If $r \geq 6$, then we add $(-6)(20) + (11)(11) = 1$ to $*$ to obtain $20(r-6) + 11(s+11) = n$. Hence we obtain n cents with $r-6$ 20-cent stamps and $s+11$ 11-cent stamps.

If $s \geq 9$, then we add $(5)(20) + (-9)(11) = 1$ to $(*)$ to obtain $20(r+5) + 11(s-9) = n$. Hence we obtain n cents with $r+5$ 20-cent stamps and $s-9$ 11-cent stamps. In all cases, it is possible to make n cents of postage. $\boxtimes$

3. **[2 parts, 15 points each]** For $n \geq 0$, let A_n be the set of all lists of length n with entries in $\{0, 1, 2\}$ not containing consecutive 1's or consecutive 2's. For example, $A_2 = \{00, 01, 02, 10, 12, 20, 21\}$ and $|A_2| = 7$. Let $a_n = |A_n|$.

   (a) For $n \geq 1$, let B_n be the lists in A_n ending in 0, and let C_n be the lists in A_n ending in 1 or 2. Prove that $|A_n| = 3|B_{n-1}| + 2|C_{n-1}|$ for $n \geq 2$.

~~Start Each list in B_{n-1} extends to a list in A_n in 3 ways, by appending a 0, 1, or a 2~~

Since each list in B_{n-1} ends with a 0, each list in B_{n-1} extends to a list in A_n in 3 ways, by appending a 0, a 1, or a 2.

Since each list L in C_{n-1} ends with a 1 or 2, each such list extends to a list in A_n in 2 ways, by appending a 0, or by appending the symbol in $\{1,2\}$ that differs from the last entry in L.

Since each list in A_n arises in one way as an extension of a list in A_{n-1}, it follows that $|A_n| = 3|B_{n-1}| + 2|C_{n-1}|$

   (b) Use part (a) to obtain a second-order linear homogeneous recurrence for a_n. Be sure to include base cases. Do not solve the recurrence.

Note that $\{B_n, C_n\}$ is a partition of A_n, and so $|A_n| = |B_n| + |C_n|$.

Using part (a), for $n \geq 2$, we compute

$$a_n = |A_n| = 3|B_{n-1}| + 2|C_{n-1}| = 2(|B_{n-1}| + |C_{n-1}|) + |B_{n-1}|$$
$$= 2|A_{n-1}| + |A_{n-2}| = 2a_{n-1} + a_{n-2}$$

where $|B_{n-1}| = |A_{n-2}|$ follows from the fact that each string in B_{n-1} consists of a string in A_{n-2} with a 0 appended. Since $A_0 = \{()\}$ empty string and $A_1 = \{0, 1, 2\}$, we have

$$\boxed{a_n = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 2a_{n-1} + a_{n-2} & \text{if } n \geq 2 \end{cases}}$$

~~There~~

4. **[20 points]** Consider a party with $2k$ people, where $k \geq 2$. Each pair of people shakes hands at most once, and every person shakes hands with an even number of people. Show that there are three people at the party who shake hands the same number of times.

~~Let t_1~~ Let t_j be the number of times that the j^{th} person shakes hands. ~~The number of handshakes that each perso~~

Note that each t_j is in the range $\{0, 1, 2, \ldots, 2k-1\}$. Moreover, since each t_j is even, we have $t_j \in \{0, 2, 4, \ldots, 2(k-1)\}$. This set has size k. Hence, $t_1, t_2, \ldots, t_{2k}$ is a ~~set~~ list L of $2k$ numbers, each of which is contained in a set of size k. Suppose for a contradiction that no number appears at least 3 times in L. It must be that each of the k numbers in $\{0, 2, 4, \ldots, 2(k-1)\}$ appears exactly twice in L. In particular, two people shake hands zero times and two people shake hands $2(k-1)$ times. But if two people do not shake any hands, then everyone shakes hands with at most $(2k-1)-2$ people, so each entry in L is at most $2k-3$. This contradicts that $2k-2$ appears twice. $\blacksquare$