Directions: Solve the following problems. All written work must be your own. See the course syllabus for detailed rules.

1. [5.1.5] Let $c \leq b \leq a$ be non-negative integers. Give two proofs, one algebraic and the other combinatorial, for $\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c}$.

2. Recall that Pascal's identity for binomial coefficients is $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ and tells the story that all k-element subsets of a set of size n either include or omit the last element.

   Suppose a, b, c are non-negative integers summing to n. What equation tells the story that in a partition of $[n]$ into 3 labeled parts of sizes a, b, and c, the last element n belongs to one of the three parts? Explain.

3. Give a combinatorial proof of the identity $\sum_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1}$. (Hint: let A be the set of all $(k+1)$-element subsets of $[n+1]$. Group the sets in A by their maximum element. Look at, for example, $n = 5$ and $k = 2$ for insight into the general case.)

4. [5.1.22] We have a group of math majors consisting of n sophomores and n juniors. We want to form a smaller group that has a total of n students in it, but from among that group we want to designate one of the students to serve as a departmental liaison. The liaison needs to be a junior, but there is no other restriction on the students chosen for the smaller group.

   (a) In how many different ways can we form the smaller group with a junior liaison?

   (b) Let k be a positive integer. In how many ways can we pick k juniors, a liaison from among the k juniors, and $n - k$ sophomores?

   (c) Use parts (a) and (b) to give a simple expression for $\sum_{k=0}^{n} k\binom{n}{k}^2$.

5. [5.1.16] Using algebra, find and prove an identity of the form $\sum_{k=0}^{n} \frac{(2n)!}{k!^2(n-k)!^2} = \binom{?}{n}^2$. (Hint: in the terms on the LHS, multiply the numerator and denominator by $n!^2$.)

6. [5.1.30] Recall that $k! \geq \left(\frac{k}{e}\right)^k$. Use this to prove that for $1 \leq k \leq n$, we have $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$.