Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[6 parts, 4 points each]** For each of the following, decide whether the given string of symbols forms a sentence. In the case of a sentence, state whether the sentence is true or false. Indicate your answer by writing the entire word "true" or "false".

P : 9 is a square number $\qquad$ $Q(x)$: x is even $\qquad$ $S(A)$: A is a finite set

(a) $P \sim \vee\, Q(4)$

Not a sentence

(b) $\sim (P \vee Q(5))$ $\quad$ P: true $\quad$ Q(5): false

$\sim(T \vee F) \equiv \sim T \equiv F$ $\quad$ This is a false sentence

(c) $S(\mathbb{R}) \implies Q(1)$

S(ℝ): false $\quad$ Q(1): false

$F \Rightarrow F \equiv T$ $\quad$ This is a true sentence

(d) $\sim (\sim (\sim P))$ $\quad$ P: True

$\sim(\sim(\sim T)) \equiv \sim(\sim F) \equiv \sim(T) \equiv F$

This is a false sentence.

(e) $(\underbrace{Q(2) \cap \mathbb{Z}}) \wedge (1 + 2 = 6)$

TRUE ∩ set $\quad$ This is not a sentence.

not a set

(f) $\forall A \in \mathcal{P}(\mathbb{N}), [A \neq \varnothing \implies \sim S(A)]$

A nonempty ⟹ A is infinite.

This is a false sentence. For example, let $A = \{1,2,3\}$. Now $A \subseteq \mathbb{N}$ and so $A \in \mathcal{P}(\mathbb{N})$. Note that $A \neq \varnothing$ and A is finite, so $\sim S(A)$ fails.

2. **[2 parts, 4 points each]** Translate the following to symbolic logic as directly and simply as possible. Indicate whether the statement is true or false by writing the entire word.

   (a) There is an integer which is within distance 1/2 to every real number.

   $\boxed{\exists a \in \mathbb{Z}, \forall x \in \mathbb{R}, \ |x-a| \leq \frac{1}{2}}$. This is $\boxed{\text{false}}$.

   (b) There are real numbers a and b such that $ax = b$ for every real number x.

   $\boxed{\exists a \in \mathbb{R}, \exists b \in \mathbb{R}, \forall x \in \mathbb{R}, \ ax = b.}$ This is $\boxed{\text{true}}$, since we may take $a = b = 0$.

3. **[2 parts, 4 points each]** Translate the following statements to English as simply as possible. Indicate whether the statement is true or false by writing the entire word.

   (a) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, [(x = 2y) \vee (x + 1 = 2y)]$

   Every consecutive pair of integers contains an even number. This is $\boxed{\text{true}}$.

   (b) $\forall S \subseteq \mathbb{R}, [(\forall n \in \mathbb{N}, |S| \geq n)) \implies (\exists x \in S, x \notin \mathbb{Q})]$

   Every infinite set of real numbers contains an irrational number.
   This is $\boxed{\text{false}}$; for example, $S = \mathbb{Z}$ or $S = \mathbb{Q}$.

4. **[2 parts, 4 points each]** For each statement below, give the logical negation in a simple, natural English statement.

   (a) For all integers a and b, if $a^2 \mid b^2$, then $a \mid b$.

   There exist integers a and b such that $a^2 \mid b^2$ but $a \nmid b$.

   (b) For each continuous function $f(x)$ such that $f(-1) < 0$ and $f(1) > 0$, there is a real number a such that $-1 < a < 1$ and $f(a) = 0$.

   There exists a continuous function $f(x)$ such that $f(-1) < 0$, $f(1) > 0$, and for each real number $a \in (-1, 1)$, we have $f(a) \neq 0$.

5. Equivalent statements.

   (a) [**8 points**] Give a truth table for φ, where φ is $(P \implies Q) \iff (Q \implies P)$.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

   (b) [**4 points**] Use the truth table to give a simple statement that is equivalent to φ.

$$\boxed{P \Leftrightarrow Q}$$

6. [**2 parts, 8 points each**] Critique the following proofs. Is the proof correct? If so, can it be improved? If not, where does the proof go wrong? Can it be fixed by modifying the proof, modifying the statement of the theorem, or both?

Theorem 1. *If $x \in \mathbb{R}$ and $x^5 > 1$, then $x > 1$.*

   (a) **Proof:** Since $x^5 > 1$, we have that $x^5 - 1 > 0$. The left hand side factors as $x^5 - 1 = (x-1)(x^4+x^3+x^2+x+1)$, giving $(x-1)(x^4+x^3+x^2+x+1) > 0$. This implies both $x - 1 > 0$ and $x^4+x^3+x^2+x+1 > 0$. □

Incorrect proof. Just because we know $A \cdot B > 0$, does not mean $A > 0$ and $B > 0$. It could be that $A < 0$ and $B < 0$.

It is not clear how to fix this argument.

   (b) **Proof:** Note that if $x > 1$, then we may multiply both sides of the inequality by the positive number x to get $x^2 > x$. Since $x^2 > x$ and $x > 1$, this implies $x^2 > 1$. Next, we multiply both sides of $x^2 > 1$ by the positive number x to obtain $x^3 > x$, and with $x > 1$ it follows that $x^3 > 1$. Continuing in this way, we see that $x^5 > 1$. Therefore $x^5 > 1$ and $x > 1$ are equivalent statements. Since we assume the hypothesis $x^5 > 1$, the equivalent statement $x > 1$ follows. □

Incorrect proof. The argument shows $x > 1$ implies $x^5 > 1$, but the converse has not been established.(*) So the claim that $x^5 > 1$ and $x > 1$ are equivalent statements has not been established. We can perhaps try to fix the argument by showing $(x^5 > 1) \Rightarrow (x^4 > 1) \Rightarrow \cdots \Rightarrow (x > 1)$.

(*). Also, the use of the phrase "Continuing in this way" is a bit vague and possibly unacceptable, but this error could be fixed.

7. [**12 points**] Prove that if x and y are odd integers, then xy is odd.

Since x and y are odd integers, we have $x = 2k+1$ and $y = 2l+1$ for some $k, l \in \mathbb{Z}$. We compute

$$xy = (2k+1)(2l+1)$$
$$= 4kl + 2l + 2k + 1$$
$$= 2(2kl + l + k) + 1.$$

Since $xy = 2a+1$ where a is the integer $2kl + l + k$, it follows that xy is odd. ∎

8. [**12 points**] Prove that if n is an even integer, then $n^2 + 2n$ is a multiple of 8. (Hint: consider appropriate cases.)

Since n is even, we have that $n = 2a$ for some $a \in \mathbb{Z}$. We consider two cases, depending on the parity of a.

Case 1: If a is even, then $a = 2b$ for some $b \in \mathbb{Z}$. We compute

$$n^2 + 2n = n(n+2) = (2a)(2a+2) = 4a(a+1) = 8b(a+1).$$

It follows that n^2+2n is a multiple of 8, since $b(a+1)$ is an integer.

Case 2: If a is odd, then $a = 2c+1$ for some $c \in \mathbb{Z}$. We compute

$$n^2 + 2n = n(n+2) = (2a)(2a+2) = 4a(a+1) = 4a(2c+2) = 8a(c+1).$$

Again it follows that n^2+2n is a multiple of 8 since $a(c+1)$ is an integer.

In all cases, n^2+2n is a multiple of 8. ∎