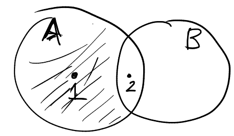
Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[2 parts, 2.5 points each]** Prove or disprove the following.



(a) If A and B are sets, then $\mathcal{P}(A-B) = \mathcal{P}(A) - \mathcal{P}(B)$.

This is false. We show that there exist sets A and B such that $\mathcal{P}(A-B) \neq \mathcal{P}(A) - \mathcal{P}(B)$. Let $A = \{1,2\}$ and $B = \{2\}$. We have

$$\mathcal{P}(A-B) = \mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}.$$

Also, we have

$$\mathcal{P}(A) - \mathcal{P}(B) = \mathcal{P}(\{1,2\}) - \mathcal{P}(\{2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\} - \{\emptyset, \{2\}\}$$
$$= \{\{1\}, \{1,2\}\}.$$

We have $\mathcal{P}(A-B) \neq \mathcal{P}(A) - \mathcal{P}(B)$. $\blacksquare$

(b) If A, B, and C are sets, then $(A \cap B) - C = (A - C) \cap (B - C)$.

This is true. We give a direct proof. Let A, B, and C be sets. First, we show $(A \cap B) - C \subseteq (A \cap C) \cap (B - C)$. Let $x \in (A \cap B) - C$. This means $x \in A \cap B$ and $x \notin C$ by definition of set minus. Since $x \in A \cap B$, we have that $x \in A$ and $x \in B$. Since $x \in A$ but $x \notin C$, we have $x \in A - C$. Similarly, since $x \in B$ and $x \notin C$, we have $x \in B - C$. Since $x \in A - C$ and $x \in B - C$, we have $x \in (A - C) \cap (B - C)$.

Secondly, we show $(A - C) \cap (B - C) \subseteq (A \cap B) - C$. Suppose $x \in (A - C) \cap (B - C)$. This means $x \in A - C$ and $x \in B - C$. Since $x \in A - C$, we have $x \in A$ but $x \notin C$. Also, since $x \in B - C$, we have $x \in B$ and $x \notin C$. Since $x \in A$ and $x \in B$, we have $x \in A \cap B$. Finally, since $x \in A \cap B$ but $x \notin C$, we have $x \in (A \cap B) - C$. $\blacksquare$

2. **[2 parts, 2.5 points each]** Prove or disprove the following.

$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

(a) If $x, y \in \mathbb{R}$ and $x^3 < y^3$, then $x < y$.

This is true. We show the contrapositive: if $x, y \in \mathbb{R}$ and $x \geq y$, then $x^3 \geq y^3$.

Suppose $x, y \in \mathbb{R}$ and $x \geq y$.

Suppose $y > 0$. Then we may divide both sides of $x \geq y$ by y to obtain $\frac{x}{y} \geq 1$. Since $\frac{x}{y} \geq 1$, we may multiply both sides by the positive number $\frac{x}{y}$ to obtain $\left(\frac{x}{y}\right)^2 \geq \frac{x}{y} \geq 1$ and again to obtain $\left(\frac{x}{y}\right)^3 \geq \frac{x}{y} \geq 1$. Hence $\frac{x^3}{y^3} \geq 1$ and multiplying both sides by y^3 gives $x^3 \geq y^3$.

So we may assume $y \leq 0$. If $x \geq 0$, then we have $y^3 \leq 0 \leq x^3$. So we may assume $y \leq x < 0$. Note that $-y \geq -x > 0$ and it follows from our previous argument that $(-y)^3 \geq (-x)^3$. So $-y^3 \geq -x^3$ or equivalently, $x^3 \geq y^3$. In all cases, $x^3 \geq y^3$. $\blacksquare$

(b) There exist integers a and b such that $42a + 63b = 3$.

This is false. We show for all integers a and b, we have $42a + 63b \neq 3$.

Let $a, b \in \mathbb{Z}$. Note that $42a + 63b = 7(6a + 9b)$, and so it follows that 7 divides $42a + 63b$. But clearly $7 \nmid 3$, and so it is not possible for $42a + 63b$ to equal 3. $\blacksquare$

Comments: 2a has shorter proofs that use more sophisticated ideas. For example:

Pf 1. Suppose $x \geq y$, and so $x - y \geq 0$. Note that

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2) = (x-y)\left(\tfrac{1}{2}(x+y)^2 + \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2\right).$$

Since both $x - y$ and $\frac{1}{2}(x+y)^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2$ are non-negative, so is their product $x^3 - y^3$. Since $x^3 - y^3 \geq 0$, we have $x^3 \geq y^3$. $\blacksquare$

Pf 2. (Sketch). Let $f(z) = z^3$ and note $f'(z) = 3z^2 \geq 0$ with equality only when $z = 0$. It follows that f is increasing on $\mathbb{R}$, and so $y \leq x$ implies $f(y) \leq f(x)$, giving $y^3 \leq x^3$. $\blacksquare$ (Note: this is just a sketch, not enough for full credit.)