Directions: You may work to solve these problems in groups, but all written work must be your own. Unless the problem indicates otherwise, all problems require some justification; a correct answer without supporting reasoning is not sufficient. See "Guidelines and advice" on the course webpage for more information.

1. Use the method of direct proof to prove the following.
   (a) Suppose a is an integer. If $7 \mid 4a$, then $7 \mid a$. Hint: the equation consider $a = 8a - 7a$.
   (b) Suppose a, b, and c are integers. If $a^2 \mid b$ and $b^3 \mid c$, then $a^6 \mid c$.
   (c) If $x \in \mathbb{R}$ and $0 < x < 4$, then $\frac{4}{x(4-x)} \geq 1$.
   (d) Every odd integer is a difference of two squares. (Example $7 = 4^2 - 3^2$).
2. Use the method of contrapositive proof to prove the following.
   (a) If n is an integer and n^2 is odd, then n is odd.
   (b) If $x \in \mathbb{R}$ and $x^3 - x > 0$, then $x > -1$.
   (c) Suppose that $x, y, z \in \mathbb{Z}$ and $x \neq 0$. If $x \nmid yz$, then $x \nmid y$ and $x \nmid z$.
   (d) If a is an integer and $4 \nmid a^2$, then a is odd.
3. Give either a direct proof or a contrapositive proof of each of the following.
   (a) If $a, b \in \mathbb{Z}$ and a and b have the same parity, then $3a + 7$ and $7b - 4$ do not.
   (b) Suppose $a, a', b, b', m \in \mathbb{Z}$ and $m \geq 1$. If $a \equiv a' \pmod{m}$ and $b \equiv b' \pmod{m}$, then $ab \equiv a'b' \pmod{m}$.
   Comment: this says that when computing ab modulo m, we are free to replace a and b with integers a' and b' of our choice, provided that a' is congruent to a and b' is congruent to b.
   (c) For all $a, b \in \mathbb{Z}$, we have $(a + b)^3 \equiv a^3 + b^3 \pmod{3}$.
   (d) If $n \in \mathbb{Z}$, then $4 \nmid (n^2 - 3)$.
4. Forced division.
   (a) Prove that for each $n \in \mathbb{N}$, there exist nonnegative integers r and s such that s is odd and $n = 2^r s$.
   (b) Show that for each $n \in \mathbb{N}$, the expression for n obtained in part (a) is unique. That is, prove that if $n = 2^{r_1} s_1$ and $n = 2^{r_2} s_2$ where r_1 and r_2 are nonnegative integers and s_1 and s_2 are odd integers, then $r_1 = r_2$ and $s_1 = s_2$.
   (c) Use part (a) to prove that if $A \subseteq \{1, 2, \ldots, 2m\}$ and $|A| > m$, then there exist integers $b, c \in A$ such that $b \mid c$.