Name: Solutions

Directions: Show all work. No credit for answers without work. Except when asked for an explicit numerical answer, you may leave answers in terms of binomial/multinomial coefficients, factorials, and sums with a small number of terms.

1. [**9 points**] A committee of 5 people must be chosen from a group of 14 employees. How many ways can the committee be chosen? Give an explicit numerical answer.

$$\binom{14}{5} = \frac{(14)_{(5)}}{5!} = \frac{14\cdot 13\cdot \cancel{12}\cdot 11\cdot \cancel{10}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 14\cdot 13\cdot 11 = (100+70+12)\cdot 11$$

$$= (182)(11) = 1820 + 182 = \boxed{2002}$$

2. [**2 parts, 8 points each**] A standard deck of cards has one card for each suit/rank pair, where the suits are spades, hearts, diamonds, and clubs, and the ranks are ace, 2 through 10, jack, queen, and king.

(a) How many ways are there to choose a set of 5 cards from the deck with at least 2 clubs?

Count Complement. $\mathcal{U}$ = all sets of 5 cards; $|\mathcal{U}| = \binom{52}{5}$

sets with 0 spades: $\binom{39}{5}$

sets with 1 spade: $\binom{13}{1}\binom{39}{4}$

At least 2 spades: $\boxed{\binom{52}{5} - \binom{39}{5} - \binom{13}{1}\binom{39}{4}} = \boxed{953{,}940}$

(b) The cards are shuffled and dealt to 4 people, with each person receiving 13 cards. What is the probability that each person's hand has exactly one king?

All distributions: $\binom{52}{13,13,13,13} = \frac{52!}{(13!)^4}$

Distributions, where each person gets one king:

① Distribute 4 kings to 4 people (4! ways)

② Distribute remaining cards $\left(\binom{48}{12,12,12,12}\text{ ways}\right)$

So $\text{Prob} = \boxed{\frac{4!\binom{48}{12,12,12,12}}{\binom{52}{13,13,13,13}}} = \frac{4!\,\frac{(48)!}{(12!)^4}}{\frac{(52)!}{(13!)^4}} = 4!\,\frac{48!}{52!}\cdot\frac{(13!)^4}{(12!)^4} = \boxed{\frac{4!\,(13)^4}{52\cdot 51\cdot 50\cdot 49}}$

3. **[3 parts, 6 points each]** How many ways are there to arrange the letters in the word ENTENTE:

   (a) without any restrictions?

   E: 3
   N: 2
   T: 2

   $$\boxed{\binom{7}{3,2,2}} = \frac{7!}{3!\cdot 2!\cdot 2!} = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{\cancel{3!}\cdot\cancel{2!\cdot 2!}} = \boxed{210}$$

   (b) so that the E's are all next to each other (as in NTEEENT)?

   single symbol: <EEE> : 1
   N: 2
   T: 2

   $$\boxed{\binom{5}{1,2,2}} = \frac{5!}{1!\cdot 2!\cdot 2!} = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{\cancel{2!\cdot 2!}} = \boxed{30}$$

   (c) so that no two E's are consecutive?

   1 Arrange N's and T's : $\binom{4}{2}$ ways
   2 Insert 3 E's into 5 gaps $\binom{5}{3}$ ways
   $\uparrow$ N $\uparrow$ T $\uparrow$ T $\uparrow$ N $\uparrow$

   $$\boxed{\binom{4}{2}\cdot\binom{5}{3}} = 6\cdot 10 = \boxed{60}$$

4. **[3 parts, 6 points each]** Count the number of non-negative integer solutions to the following.

   (a) $x_1 + \ldots + x_6 = 30$

   30 stars
   5 bars

   $$\boxed{\binom{35}{5}} = \boxed{324,632}$$

   (b) $x_1 + \ldots + x_6 = 30$, such that $x_i \geq i$ for $1 \leq i \leq 6$

   Reserve $1+2+\cdots+6$ stars: $1+2+\cdots+6 = \binom{7}{2} = \frac{7\cdot 6}{2} = 21$

   So $\hat{x}_1 + \cdots + \hat{x}_6 = 9$, $\hat{x}_i \geq 0$.

   So 9 stars, 5 bars $\Rightarrow \boxed{\binom{14}{5}} = \boxed{2002}$

   (c) $x_1 + \ldots + x_6 = 30$ such that $x_i \leq 20$ for each i.

   $\mathcal{U}$: all solns : 30 stars, 5 bars $\Rightarrow \binom{35}{5}$

   A_i: solns where $x_i \geq 21$: $\hat{x}_1 + \cdots + \hat{x}_6 = 9 \Rightarrow \binom{14}{5}$ solns.

   Note: $|A_i \cap A_j| = 0$. So # solns is $|\mathcal{U}| - 6|A_1| = \boxed{\binom{35}{5} - 6\binom{14}{5}}$

   $= \boxed{312,620}$

5. **[10 points]** Give an algebraic and combinatorial proofs of the identity $t^3 = 6\binom{t}{3} + 6\binom{t}{2} + \binom{t}{1}$.

Algebraic Proof: We compute $6\binom{t}{3} + 6\binom{t}{2} + \binom{t}{1} = 6\,\frac{t(t-1)(t-2)}{3!} + 6\,\frac{t(t-1)}{2} + t$

$= t(t-1)(t-2) + 3t(t-1) + t = t(t-1)[(t-2)+3] + t = t(t-1)(t+1) + t$

$= t[(t-1)(t+1)+1] = t[t^2-1+1] = t^3.$ ▨

Combinatorial Proof. The LHS counts triples (x_1, x_2, x_3) with each $x_i \in [t]$, repeated values are allowed. The RHS also counts these triples according to the number of values that appear in the triple. There are t such triples with 1 value, like $(2,2,2)$ or $(5,5,5)$. There are $3!\binom{t}{3}$ such triples with 3 distinct values, like $(2,3,8)$ or $(3,2,8)$: choose 3 distinct values ($\binom{t}{3}$ options), and then order those values ($3!$ options). There are $\binom{t}{2}\cdot 2\cdot 3$ such triples with 2 distinct values: choose the values ($\binom{t}{2}$ options), choose which one appears only once (2 options), and choose a position for the value appearing once (3 options). ▨

6. **[5 points]** Use the identity in the previous problem to give a formula for $\sum_{t=1}^{n} t^3$. (Hint: an identity from HW10 may be helpful; it counts the number of $(k+1)$-element subsets of $[n+1]$ by grouping the subsets by maximum value.)

We need the so-called "Hockey Stick Identity": $\sum_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1}$.

We compute $\sum_{t=1}^{n} t^3 = \sum_{t=1}^{n} 6\binom{t}{3} + 6\binom{t}{2} + \binom{t}{1}$

$= 6\sum_{t=1}^{n}\binom{t}{3} + 6\sum_{t=1}^{n}\binom{t}{2} + \sum_{t=1}^{n}\binom{t}{1}$

$= 6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2} = 6\Big[\overbrace{\binom{n+1}{4} + \binom{n+1}{3}}^{\text{Combine w/ Pascal's Identity}}\Big] + \binom{n+1}{2}$

$= \boxed{6\binom{n+2}{4} + \binom{n+1}{2}}$

7. **[8 points]** Use the binomial theorem to find the coefficient of x^7 in the expansion of $(x+1)^{20}$.

$$(x+1)^{20} = \sum_{k=0}^{20} \binom{20}{k} x^k = \dots + \binom{20}{7} x^7 + \dots$$

So coefficient is $\boxed{\binom{20}{7}}$.

8. **[2 parts, 8 points each]** Find simple formulas for the following sums.

(a) $\sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^k$

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^k (1)^{n-k} = \left(\frac{1}{2}+1\right)^n = \boxed{\left(\frac{3}{2}\right)^n}$$

(b) $\sum_{k=0}^{n} \binom{n}{k} k 2^k$ (Hint: differentiate the binomial theorem expansion for $(x+1)^n$.)

$$\frac{d}{dx}\left[(x+1)^n\right] = \frac{d}{dx}\left[\sum_{k=0}^{n} \binom{n}{k} x^k\right]$$

$$n(x+1)^{n-1} = \sum_{k=1}^{n} k \binom{n}{k} x^{k-1}$$

mult. both sides by x:

$$nx(x+1)^{n-1} = \sum_{k=1}^{n} k \binom{n}{k} x^k$$

Set $x = 2$:

$$\sum_{k=1}^{n} k \binom{n}{k} 2^k = n \cdot 2 \cdot (3)^{n-1}$$

Note: $k=0$ term contributes 0, so

$$\sum_{k=0}^{n} \binom{n}{k} k \cdot 2^k = \boxed{(2n)3^{n-1}}.$$