Directions: Solve the following problems. All written work must be your own. See the course syllabus for detailed rules.

1. Prove that for every nonnegative integer n, we have $\sum_{k=1}^{n} k^2 = [(2n+1)(n+1)n]/6$.

2. Recall that $n! = 1 \cdot 2 \cdot \cdots \cdot n$. For $n \geq 0$, find a formula for $1 + \sum_{k=1}^{n} k \cdot k!$ and prove your formula is correct.

3. The *adjusted Fibonacci sequence* $\hat{F}_n$ is given by $\hat{F}_0 = \hat{F}_1 = 1$ and $\hat{F}_n = \hat{F}_{n-1} + \hat{F}_{n-2}$ for $n \geq 2$. (Note that $\hat{F}_2 = \hat{F}_1 + \hat{F}_0 = 1 + 1 = 2$, and $\hat{F}_3 = \hat{F}_2 + \hat{F}_1 = 2 + 1 = 3$.) Prove that for $n \geq 0$, we have $\hat{F}_n \leq \phi^n$, where $\phi = (1 + \sqrt{5})/2$.

4. A *unit fraction* is a rational number of the form $1/n$ for some positive integer n. An *Egyptian fraction* is the sum of zero or more distinct unit fractions. For example, $\frac{29}{45}$ is an Egyptian fraction since $\frac{29}{45} = \frac{1}{4} + \frac{1}{5} + \frac{1}{9} + \frac{1}{12}$. Although $\frac{11}{12} = \frac{1}{3} + \frac{1}{3} + \frac{1}{4}$, this is not enough to establish that $\frac{11}{12}$ is an Egyptian fraction since the unit sums are not distinct. However, $\frac{11}{12} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6}$ does establish that $\frac{11}{12}$ is an Egyptian fraction.

   (a) Let a and b be nonnegative integers with $0 < a < b$, and let n be the smallest positive integer such that $\frac{1}{n} \leq \frac{a}{b}$. Prove that $\frac{a}{b} - \frac{1}{n} = \frac{c}{d}$ for some nonnegative integers c and d such that $c < a$ and $\frac{c}{d} < \frac{1}{n}$.

   (b) Prove that if a and b are nonnegative integers with $a < b$, then $\frac{a}{b}$ is an Egyptian fraction.

   Comment: with part (b) and the fact that the Harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges, one can show that every nonnegative rational number is an Egyptian fraction.