Directions: Solve 4 of the following 5 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. Let $(X, \mathcal{F})$ be a Steiner Triple System with $|X| = 7$.
   (a) Prove that if $A, B \in \mathcal{F}$, then $|A \cap B| = 1$.
   (b) Prove that, up to relabeling the points in X, there is only one Steiner Triple System of order 7.
2. Prove that for each $k \in \mathbb{N}$ there exists n such that in each k-coloring of the non-empty subsets of $[n]$ has disjoint sets A and B such that A, B, and $A \cup B$ have the same color. (Hint: show that $n \leq R_k(3; 2)$, where $R_k(3, 3, \ldots, 3)$ is the minimum number of vertices in a complete graph in which every k-coloring contains a monochromatic triangle.)
3. Let S be a set of $R(m, m; 3)$ points in the plane no three of which are collinear. Prove that S contains m points that form a convex m-gon.
4. The graph mK_2 is the graph on $2m$ vertices consisting of m disjoint copies of K_2. Prove that $R(mK_2, mK_2) = 3m - 1$.
5. A function $f \colon [n] \to [n]$ is **contractive** if $f(i) \leq i$ for all i. A **monotone k-list** for f is a strictly increasing list $a_1, \ldots, a_k$ from $[n]$ such that $f(a_1) \leq \cdots \leq f(a_k)$. Prove that 2^{k-1} is the least n such that for every contractive mapping on $[n]$ there is a monotone k-list. (Note: there are 2 things to prove. First, you must provide an example of a contractive function $f \colon [n] \to [n]$ for $n = 2^{k-1} - 1$ that has no monotone k-list. Next, you must show that every contractive function $f \colon [n] \to [n]$ with $n \geq 2^{k-1}$ has a monotone k-list.) (Hint: For how many a in $[n]$ can the longest monotone list ending with a have j elements?)