Directions: Solve 5 of the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines that apply to all homeworks.

1. Show that a deck of 16 cards with ranks $\{1, 2, 3, 4\}$ and suits $\{a, b, c, d\}$ can be arranged in a 4×4 array so that each row and each column contains each rank and each suit exactly once.

2. Show that it is possible for 9 students to attend a 4 day orientation, with all students walking to campus in groups of three and each pair of students walking together on exactly one day.

3. Recall that $[n] = \{1, 2, \ldots, n\}$.

   (a) Count the subsets of $[n]$ that contain at least one odd integer.

   (b) Count the k-sets in $[n]$ having no two consecutive integers.

   (c) Count the lists of subsets $A_0, A_1, \ldots, A_n$ of $[n]$ such that $A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n$.

   (d) Count the lists such that $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n$.

4. Count the lists of m ones and n zeros that have exactly k runs of ones, where a **run** is a maximal set of consecutive entries with the same value.

5. A permutation is **graceful** if the absolute differences between successive elements are distinct. Prove that if the set of elements in even-indexed positions of a graceful permutation of $[2n]$ is $[n]$, then the first and last elements differ by n. (*Hint*: Let π be a graceful permutation of $[2n]$ such that $\pi(i) \leq n$ if and only if i is even, and evaluate $|\pi(2n) - \pi(1)| + \sum_{i=1}^{2n-1} |\pi(i) - \pi(i+1)|$ in two different ways).

6. The **displacement** of a permutation π of $[n]$ is $\sum_{i=1}^{n} |\pi(i) - i|$. Note that the displacement of π is zero if and only if π is the identity permutation.

   (a) For each n, give an example of a permutation of $[n]$ with displacement $\lfloor n^2/2 \rfloor$.

   (b) Let π be a permutation of $[n]$, let $S = \{(i, \pi(i)) \colon i \in [n]\}$, let $A = \{(x, y) \in S \colon y \geq x\}$, and let $B = \{(x, y) \in S \colon y < x\}$. (Think of A as the set of points in the graph of π on or above the line $y = x$ and B as the set of points in the graph of π below the line $y = x$.) Prove that if π is a permutation with maximum displacement, $(x, y) \in A$, and $(x', y') \in B$, then $x < x'$ and $y > y'$.

   (c) Use part (b) to show that every permutation of $[n]$ has displacement at most $\lfloor n^2/2 \rfloor$.

   (d) For even n, count the number of permutations of $[n]$ that have maximum displacement. (Remark: the analysis above also makes it possible to count the maximum displacement permutations for general n, but the computation is not as nice when n is odd.)