Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[12 points]** Suppose that $a \in \mathbb{Z}$. Prove that if a is odd, then $4 \mid a^2 + 2a - 3$.

Suppose that a is odd, and so $a = 2k+1$ for some $k \in \mathbb{Z}$. We compute

$$\begin{aligned} a^2 + 2a - 3 &= (2k+1)^2 + 2(2k+1) - 3 \\ &= 4k^2 + 4k + 1 + 4k + 2 - 3 \\ &= 4(k^2 + k + k) = 4(k^2 + 2k). \end{aligned}$$

Since $k^2 + 2k \in \mathbb{Z}$, we conclude that $4 \mid a^2 + 2a - 3$. ∎

2. **[12 points]** Suppose that $a \in \mathbb{Z}$. Prove that if $3 \mid a^2 + 5a + 1$, then $3 \nmid a$.

We ~~show~~ the contrapositive: if $3 \mid a$, then $3 \nmid a^2 + 5a + 1$.

Suppose that $3 \mid a$, and so $a = 3k$ for some $k \in \mathbb{Z}$. We compute $a^2 + 5a + 1 = (3k)^2 + 5(3k) + 1 = 9k^2 + 15k + 1 = 3(3k^2 + 5k) + 1$.

Dividing by 3 gives $\frac{a^2+5a+1}{3} = (3k^2 + 5k) + \frac{1}{3}$. Since $3k^2 + 5k \in \mathbb{Z}$, it follows that $(3k^2 + 5k) + \frac{1}{3}$ is not an integer. Since $\frac{a^2+5a+1}{3} \notin$

we have that $3 \nmid a^2 + 5a + 1$. ∎

3. **[2 parts, 12 points each]** Irrational numbers.

(a) Prove that $\sqrt{6}$ is irrational.

Suppose for a contradiction that $\sqrt{6}$ is rational, and so $\sqrt{6} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. We may assume without loss of generality that a and b have no common positive divisors besides 1. Squaring both sides and rearranging gives $a^2 = 6b^2 = 2(3b^2)$. Since a^2 is even and an odd number times an odd number is odd, it follows that a is even. So $a = 2k$ for some $k \in \mathbb{Z}$, and $(2k)^2 = 2(3b^2)$ becomes $2k^2 = 3b^2$. Since $3b^2$ is even and the product of three odd integers is odd, it follows that b is even. Therefore a and b are both even. But this contradicts that a and b have no common divisors besides 1. ☒

(b) Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Suppose for a contradiction that $\sqrt{2} + \sqrt{3}$ is rational, and so $\sqrt{2} + \sqrt{3} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Squaring both sides gives $\frac{a^2}{b^2} = (\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$.

SPACE

Solving for $\sqrt{6}$, we have $\sqrt{6} = \frac{1}{2}\left(\frac{a^2}{b^2} - 5\right) = \frac{1}{2}\left(\frac{a^2 - 5b^2}{b^2}\right) = \frac{a^2 - 5b^2}{2b^2}$.

It follows that $\sqrt{6}$ is rational, which contradicts part (a). ☒

4. **[12 points]** Determine the coefficient of x^3 in $(x+1)^8 - (x-1)^6$ explicitly. No justification/sentences equ

We have $(x+1)^8 = \sum_{k=0}^{8} \binom{8}{k} x^k (1)^{8-k} = \sum_{k=0}^{8} \binom{8}{k} x^k$ and $(x-1)^6 = \sum_{k=0}^{6} \binom{6}{k} x^k (-1)^{6-k}$. The coefficients of x^3 in both cases is given by the $k=3$ term. Therefore the coefficient of x^3 in $(x+1)^8$ is $\binom{8}{3}$ and in $(x-1)^6$ is $\binom{6}{3}(-1)^{6-3}$

Therefore the coefficient of x^3 in $(x+1)^8 - (x-1)^6$ is given by

$$\binom{8}{3} - \left((-1)\binom{6}{3}\right) = \binom{8}{3} + \binom{6}{3} = \frac{8\cdot 7\cdot \cancel{6}\cdot \cancel{5!}}{\cancel{3!}\cdot \cancel{5!}} + \frac{\cancel{6}\cdot 5\cdot 4\cdot \cancel{3!}}{\cancel{3!}\cdot \cancel{3!}} = 56+2$$

$$= \boxed{76}.$$

5. **[12 points]** Suppose that $a, b \in \mathbb{Z}$. Use the binomial theorem to prove that $(a+b)^5 \equiv a^5 + b^5 \pmod 5$.

By the Binomial Theorem, we have

$$(a+b)^5 = \sum_{k=0}^{5} \binom{5}{k} a^k b^{5-k} = b^5 + \binom{5}{1} ab^4 + \binom{5}{2} a^2 b^3 + \binom{5}{3} a^3 b^2 + \binom{5}{4} a^4 b$$

$$= b^5 + 5ab^4 + 10a^2b^3 + 10a^3b^2 + 5a^4b + a^5.$$

It follows that

$$(a+b)^5 - (a^5+b^5) = 5\left(ab^4 + 2a^2b^3 + 2a^3b^2 + a^4b\right)$$

and so $5 \mid (a+b)^5 - (a^5+b^5)$. It follows that $(a+b)^5 \equiv a^5 + b^5 \pmod 5$.

6. [**12 points**] Critique the following proof. Is it correct? If so, can it be improved? If not, can it be fixed by modifying the proof, modifying the statement of the theorem, or both?

Theorem 1. *If $x, y \in \mathbb{Z}$, then $x^3 + y^3$ is not prime.*

Proof: Observe that $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$. Since $x + y$ divides $x^3 + y^3$ and $1 < x + y < x^3 + y^3$, it follows that $x^3 + y^3$ is not prime. $\square$

This proof is not correct. ~~There are two~~ The problem is that, since x and y are arbitrary integers, it need not be true that $1 < x+y < x^3+y^3$. For example, if $x = 3$ and $y = -2$, then we have $x+y = 1$ and $x^3+y^3 = 27+(-8) = 19 = 1 \cdot 19$. ~~$= 1(3+(-2))$~~

Continuing, $1 \cdot 19 = (x+y)(x^2-xy+y^2)$. We can fix this problem by requiring that x and y are both positive integers; that is, by assuming $x, y \in \mathbb{N}$. In this case, $x+y \geq 1+1 = 2 > 1$. However, we still may have $x+y = x^3+y^3$ if $x = y = 1$. As long as x and y are not both 1, and $x, y \in \mathbb{N}$, it is the case that $x+y < x^3+y^3$. So we may fix the proof by modifying the statement of the problem:

<u>Theorem</u> If x and y are positive integers and they are not both 1, then x^3+y^3 is not prime.

7. **[16 points]** Suppose that n is an integer. Prove that n is the product of two consecutive integers if and only if $4n+1$ is an odd square.

($\Rightarrow$) Suppose that n is the product of two consecutive integers. Therefore $n = a(a+1)$ for some $a \in \mathbb{Z}$. We show that $4n+1$ is an odd square. Indeed,

$$4n+1 = 4[a(a+1)] + 1 = 4a^2 + 4a + 1 = (2a+1)^2.$$

Since $2a+1$ is odd, it follows that $4n+1$ is an odd square.

($\Leftarrow$) Conversely, suppose that $4n+1$ is an odd square, and so $4n+1 = (2t+1)^2$ for some $t \in \mathbb{Z}$. We have $4n+1 = 4t^2 + 4t + 1$ and solving for n gives $n = t^2 + t = t(t+1)$. Therefore n is the product of the consecutive integers t and $t+1$.

(Scratch Paper)