Directions: You may work to solve these problems in groups, but all written work must be your own. Unless the problem indicates otherwise, all problems require some justification; a correct answer without supporting reasoning is not sufficient. Submissions must be stapled. See "Guidelines and advice" on the course webpage for more information.

1. Prove the following using the method of proof by contradiction.
   (a) Show that $2^{\frac{1}{3}}$ is irrational.
   (b) Suppose that $a, b, c \in \mathbb{Z}$. Show that if $a^2 + b^2 = c^2$, then a or b is even.
   (c) Prove that there are no integers a and b such that $21a + 30b = 1$.
2. Irrational powers of three.
   (a) Let a be an integer. Prove that if $3 \mid a^2$, then $3 \mid a$.
   (b) Prove that if k is an odd positive integer, then $\sqrt{3^k}$ is irrational. Hint: suppose for a contradiction that the implication is false for some values of k, and let k be the least odd positive integer for which the implication fails.
3. Identities.
   (a) Give a combinatorial proof of the identity $\sum_{k=1}^{n} k(n-k) = \binom{n+1}{3}$.
   (b) Use part (a) to derive a closed-form formula for $\sum_{k=1}^{n} k^2$.
4. Using only logic and trigonometry (not calculus), show that $\sin(x) + \sqrt{3}\cos(x) \leq 2$ for each real number x. (Hint: recall that $\tan(\pi/3) = \sqrt{3}$.)
5. Critique the following argument. (Be careful!)

   Theorem 1. *If $p_1, \ldots, p_k$ is a list of the first k primes, then $p_1 p_2 \cdots p_k + 1$ is also a prime.*

   Proof: Let $n = p_1 p_2 \cdots p_k + 1$, and note that $1 = n - p_1 p_2 \cdots p_k$.
   Suppose for a contradiction that some prime p_i less than n divides n. If this were true, then p_i divides both terms on the right hand side of $1 = n - p_1 p_2 \cdots p_k$ and therefore p_i must also divide the left hand side of this equation. Since no prime divides 1, we have a contradiction.
   The contradiction implies that no prime less than n divides n, and therefore n is prime. $\square$