Directions: Solve 5 of the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. Let G be a t-vertex m-edge graph with $m \geq 2$. Show that $\mathrm{ex}(n, G) \geq cn^{2-\frac{t-2}{m-1}}$. Compare this lower bound in the case that $G = C_{2k}$ with the Bondy–Simonovits theorem.

2. *Turán number of* C_6.

   (a) Prove that if G is an m-edge graph with no 6-cycle, then G has a subgraph with at least $m/4$ edges and girth at least 8.

   (b) Use part (a) to show that $\mathrm{ex}(n, C_6) \leq cn^{4/3}$ for some constant c.

3. *The diamond poset Turán problem.*

   (a) Let $\mathcal{F} \subseteq 2^{[n]}$ be nonempty. For each $A \in \mathcal{F}$, let I_A be the number of times that a random chain from A to $[n]$ meets $\mathcal{F}$. (Note that since $A \in \mathcal{F}$, always $I_A \geq 1$.) Show that there exists $A \in \mathcal{F}$ such that $\mathbb{E}(I_A) \geq \ell(\mathcal{F})$.

   (b) Prove that if $\ell(\mathcal{F}) > 2.5$, then $\mathcal{F}$ weakly contains the diamond poset $2^{[2]}$. Conclude that $\mathrm{La}(n, 2^{[2]}) \leq 2.5\binom{n}{n/2}$.

4. The t-dimensional hypercube, denoted Q_t, has vertex set $\{0,1\}^t$ with vertices adjacent if and only if they disagree in exactly one coordinate. Prove that there exists a constant c such that $R(Q_t, Q_t) \leq 2^{ct}$ for all t. (Hint: given a 2-edge-coloring of K_n, apply a modified the dependent random choice lemma to a monochromatic subgraph with density at least 1/2.)

5. In a hypergraph, the *degree* of a set of vertices S, denoted $d(S)$, is the number of edges containing S. Let $n \geq 5$ and let G be an n-vertex 3-uniform hypergraph such that $d(S) = d(S') > 0$ when $|S| = |S'| = 2$. Prove that $\chi(G) > 2$.

6. Let G be the 3-uniform complete 3-partite graph with t vertices in each part.

   (a) Let H be an n-vertex 3-uniform graph. For a set $S \subseteq V(H)$, let $d(S)$ be the number of edges in H that contain S. Prove that if $\sum_{S \in \binom{V(H)}{2}} \binom{d(S)}{t} > \mathrm{ex}(n, K_{t,t})\binom{n}{t}$, then $G \subseteq H$.

   (b) Prove that $\mathrm{ex}(n, G) \leq c_t n^{3-\frac{1}{t^2}}$ for some constant c_t.