Directions: Solve 5 of the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. A *chord* of a cycle C is an edge $e \in E(G) - E(C)$ that has both endpoints on C.
   (a) Show that if $\delta(G) \geq 3$, then G has a cycle with a chord.
   (b) Prove that for $n \geq 4$, if G has at least $2n - 3$ edges, then G has a cycle with a chord.

2. Prove that an n-vertex graph with m edges has at least $\frac{m}{3n}(4m - n^2)$ triangles. (Hint: adapt a proof form class.)

3. Recall that a *decomposition* of a graph G is a list of subgraphs $H_1, \ldots, H_t$ such that each edge in G appears in exactly one of $\{H_1, \ldots, H_t\}$; the *size* of the decomposition $H_1, \ldots, H_t$ is t.
   (a) Prove that each graph G has a matching of size at least $\delta(G)/2$.
   (b) Use part (a) to show that each n-vertex graph has a decomposition into triangles and edges of size at most $n^2/4$. (Hint: first handle edges incident to a vertex u of minimum degree, and then apply induction to an appropriate subgraph of $G - u$.)

4. For a vertex v in an n-vertex graph G, let $f(v) = \alpha(G[N(v)])$; that is, $f(v)$ is the maximum size of an independent set in the neighbors of v. Prove that $\sum_{v \in V(G)} f(v) \leq \lfloor n^2/2 \rfloor$ and determine which graphs achieve equality.

5. Let G be an n-vertex graph and let S be the set of nonnegative vectors in $\mathbb{R}^n$ that sum to 1. Given an n-vertex graph G, define $f(x) = \sum_{uv \in E(G)} x_u x_v$, and let $M = \max_{x \in S} f(x)$.
   (a) Prove that $M = \frac{1}{2}(1 - \frac{1}{r})$, where $r = \omega(G)$. (Recall that $\omega(G)$ is the maximum size of a clique in G.) (Hint: if $uv \notin E(G)$, then show that $f(x') \geq f(x)$ for some x' with $x_u x_v = 0$.)
   (b) Show that $|E(G)| \leq \frac{n^2}{2}(1 - \frac{1}{r})$.

   Comment: it is also possible to derive the structure of extremal examples from this proof with some additional work. Start by showing that M is attained by an $x \in S$ with all positive coordinates if and only if G is a complete r-partite graph.

6. Let G be a connected graph such that $d(u) + d(v) \geq k$ whenever $uv \notin E(G)$. Prove that G is Hamiltonian or has a copy of P_{k+1}.