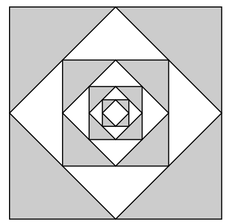
Name: ______________________________

Directions: All questions require explanation in English sentences.

1. **[10 points]** The midpoints of the sides of a square are joined to form another square, and this process is repeated. The outer square has side length 1. What is the total area of the shaded regions?



2. **[2 parts, 5 points each]** Consider the following argument.

   Theorem 1. *If n is an integer and $n \geq 5$, then $n^2 - 16$ is not prime.*

   Proof: Using algebra, we see that $n^2 - 16 = (n+4)(n-4)$. Since $n+4$ divides $n^2 - 16$, we conclude that $n^2 - 16$ is not prime. $\square$

   (a) Execute the proof firstly for $n = 5$ and secondly for $n = 6$.

   (b) Analyze the proof above. Is it a valid proof? If not, can it be corrected? If possible, how would you correct it?

3. **[2 parts, 5 points each]** Consider the following argument.

> **Theorem 2.** *If a and b are nonnegative real numbers, then $(a+b)/2 \geq \sqrt{ab}$.*
>
> **Proof:** Since the square of a real number is nonnegative, we have $(a-b)^2 \geq 0$. Expanding the left hand side, we obtain $a^2 - 2ab + b^2 \geq 0$. Adding $4ab$ to both sides, we see that $a^2+2ab+b^2 \geq 4ab$, or $(a+b)^2 \geq (2\sqrt{ab})^2$. Since $a+b \geq 0$ and $2\sqrt{ab} \geq 0$, we may take the square root of both sides, obtaining $a+b \geq 2\sqrt{ab}$. Dividing both sides by 2, we conclude $(a+b)/2 \geq \sqrt{ab}$. □

(a) Execute the proof for $a = 3$ and $b = 5$.

(b) Analyze the proof above. Is it a valid proof? If not, can it be corrected? If possible, how would you correct it?

4. [**5 points**] One of the following implications is true and the other is false. Identify which is which. Prove the true implication and find a counterexample for the other. Let a be a real number.

   - If a^2 is irrational, then a is irrational.
   - If a is irrational, then a^2 is irrational.

5. [**5 points**] For which real values of a is the polynomial $x + a$ a factor of $x^3 + 3ax^2 - a$?

6. **[4 parts, 2.5 points each]** Let (*) be the equation $3x^2 + (x-1)y = 4$. Decide whether the following statements are true or false. Explain your answer.

   (a) For each real number x and each real number y, the pair x, y satisfies (*).

   (b) There exists a real number x such that for each real number y, the pair x, y satisfies (*).

   (c) For each real number x, there exists a real number y such that the pair x, y satisfies (*).

   (d) For each real number y, there exists a real number x such that the pair x, y is satisfies (*).

7. **[10 points]** Let f and g be polynomials of degree at most n, and suppose that $a_1, \ldots, a_{n+1}$ are distinct real numbers such that $f(a_i) = g(a_i)$ for each i. Prove that $f = g$. Hint: let $h(x) = f(x) - g(x)$. What can you say about the degree of h?