**Directions:** Solve 5 of the following 6 problems. All written work must be your own, and you must cite any external sources that you use.

1. [CM 15.1.9] Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be families of subsets of $[n]$ such that $|A_i \cap B_i|$ is odd for all $i$ and $|A_i \cap B_j|$ is even when $i \neq j$. Prove that $m \leq n$.

2. [CM 15.1.8] Let $A_1, \ldots, A_m$ be a family of even-sized subsets of $[n]$ such that all pairwise intersections have odd size.

   (a) Prove that $m \leq n$, with equality possible when $n$ is odd. Hint: first try to prove that $m \leq n + 1$ using standard techniques. Then, add a new polynomial to the independent set that does not increase dimension.

   (b) Prove that $m \leq n - 1$ when $n$ is even, with equality possible. Hint: suppose for a contradiction that $m = n$ and $n$ is even.

3. [CM 15.1.23] 3-Choosability of planar bipartite graphs.

   (a) The *maximum density* $\rho(G)$ of a graph $G$ is $\max_{H \subseteq G} |E(H)|/|V(H)|$. Prove that if $\rho(G) \leq d$, then $G$ has an orientation with outdegree at most $d$.

   (b) Conclude from part (a) that planar bipartite graphs are 3-choosable.

4. [CM 15.1.29] Use the Combinatorial Nullstellensatz to prove that the minimum number of hyperplanes in $\mathbb{R}^n$ that do not contain 0 but together cover all the other points in $\{0,1\}^n$ is $n$. (Recall that a hyperplane in $\mathbb{R}^n$ is a set of points of the form $\{x \in \mathbb{R}^n \colon \langle x, v \rangle = c\}$ for some $c \in \mathbb{R}$ and $v \in \mathbb{R}^n$.)

5. [CM 15.2.23] The Permanent Lemma.

   (a) Let $A$ be an $n$-by-$n$ matrix with nonzero permanent over a field $\mathbb{F}$. Use the Combinatorial Nullstellensatz to prove that for any $b \in \mathbb{F}^n$ and sets $S_1, \ldots, S_n$ of size 2 in $\mathbb{F}$, there is a vector $x \in \prod_{i=1}^n S_i$ such that $Ax$ differs from $b$ in every coordinate.

   (b) Let $p$ be a prime. Prove that every list of $2p - 1$ members of $\mathbb{Z}_p$ contains $p$ entries that sum to 0 modulo $p$.

6. [CM 15.3.16] Let $B_1, \ldots, B_m$ be a bliclique decomposition of the clique with vertex set $[n]$, with $B_k$ having partite sets $X_k$, $Y_k$. Let $A_k$ be the $0,1$-matrix having 1 in position $(i, j)$ if and only if $i \in X_k$ and $j \in Y_k$. Let $S = \sum_{k=1}^m A_k$.

   Observe that $S + S^T = J - I$. Prove that every $n$-by-$n$-matrix satisfying this equation has rank at least $n - 1$. Since $\operatorname{rank}(A + A') \leq \operatorname{rank}(A) + \operatorname{rank}(A')$ for all matrices $A$ and $A'$, conclude $\operatorname{rank} S \leq m$ and therefore $m \geq n - 1$.

   Hint: to show $\operatorname{rank} S \geq n - 1$, assume for a contradiction that $\operatorname{rank} S \leq n - 2$, add observe that adding any additional row gives an $(n+1)$-by-$n$ matrix $S'$ with rank less than $n$, so that $S'x = 0$ has a nontrivial solution.