Directions: Solve 5 of the following 6 problems. All written work must be your own, and you must cite any external sources that you use.

1. [AS 2.9] Let G be a bipartite graph with n vertices let L be a list-assignment with $|L(v)| > \lg n$ for each vertex v. Prove that G has a proper L-coloring.

2. [AS 3.3] Let G be an n-vertex 3-uniform hypergraph with m edges. Prove that if $m \geq n/3$, then $\alpha(G) \geq \frac{2n^{2/3}}{3\sqrt{3m}}$. (In a hypergraph, a set S is an independent set if no edge has all its vertices in S.)

3. Let $p = \frac{c \ln n}{n}$ where c is a constant, and let X be the number of isolated vertices in $G_{n,p}$.

   (a) Use the first moment method to show that if $c > 1$, then $X = 0$ with high probability.

   (b) Use the second moment method to show that if $c < 1$, then $X \geq 1$ with high probability.

   Comment: the function $p(n) = \frac{\ln n}{n}$ is a *threshold* for the disappearance of isolated vertices. That is, as $G_{n,p}$ evolves from an empty graph ($p = 0$) to a complete graph ($p = 1$), the isolated vertices disappear at $p = \frac{\ln n}{n}$.

4. [AS 4.4] Let X be a random variable, and let A be an event. Let Y be the random variable which has value $\mathbb{E}(X|A)$ if A occurs and value X otherwise.

   (a) Prove that $\mathbb{E}(Y) = \mathbb{E}(X)$.

   (b) Define the conditional variance $\mathrm{Var}(X|A)$ to be the variance of X in the restriction of Ω to A; that is $\mathrm{Var}(X|A) = \mathbb{E}(X^2|A) - (\mathbb{E}(X|A))^2$. Prove that $\mathrm{Var}(Y) = \mathrm{Var}(X) - \mathrm{Var}(X|A) \cdot \Pr(A)$. Conclude that $\mathrm{Var}(Y) \leq \mathrm{Var}(X)$.

   (c) Use part (b) to show that if $\mathbb{E}(X) = 0$ and $\sigma^2 = \mathrm{Var}(X)$, then $\Pr(X \geq \lambda) \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}$ when $\lambda > 0$.

5. [MR 4.3] Let G be a graph with maximum degree k, and let L be a list assignment. For each vertex v, let w_v be a non-negative weight function $w_v \colon L(v) \to \mathbb{R}$ such that $\sum_{c \in L(v)} w_v(c) = 1$ for each vertex v. Show that if $\sum_{c \in L(u) \cap L(v)} w_u(c) w_v(c) \leq 1/(8k)$ for each edge $uv \in E(G)$, then G has a proper L-coloring.

6. [CM 14.2.16] Let G be a digraph in which every vertex has outdegree k and indegree k, and let $r = \lfloor k/(2 + 2\ln k) \rfloor$. Partition $V(G)$ into r nonempty sets $V_1, \ldots, V_r$ by an appropriate experiment. Use the Local Lemma to prove that with positive probability every vertex has a successor in the set containing it. Conclude that every k-regular directed graph has a family of r-pairwise disjoint cycles. (Hint: be careful to ensure that $V_1, \ldots, V_r$ are nonempty.)