Directions: Solve 5 of the following 6 problems. All written work must be your own, and you must cite any external sources that you use.

1. (a) Recall that $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d(v)+1}$. Use this to show that if G is an n-vertex graph with m edges, then $\alpha(G) \geq \frac{n^2}{2m+n}$.

   (b) Let X and Y be chosen independently from $\{1, \ldots, n\}$ according to the same distribution. (The distribution is not necessarily uniform.) Prove that $\Pr(X = Y) \geq 1/n$.

2. The outdegree of a vertex v in a directed graph is denoted by $d^+(v)$.

   (a) Give a short probabilistic proof that for each n, there exists an n-vertex tournament with at least $\frac{1}{4}\binom{n}{3}$ directed triangles.

   (b) Prove that the number of directed triangles in an n-vertex tournament T is exactly $\binom{n}{3} - \sum_{v \in V(T)} \binom{d^+(v)}{2}$.

   (c) Prove that every n-vertex tournament has at most $\frac{1}{4}\binom{n+1}{3}$ directed triangles.

3. [AS 1.4] Let G be a graph with n vertices and minimum degree k. Prove that there is a partition of $V(G)$ into two parts A and B such that $|A| \leq O\left(\frac{\log k}{k} n\right)$ and every vertex in B has at least one neighbor in A and at least one neighbor in B.

4. Let S be a collection of binary strings, and let n_k be the number of strings in S of length k.

   (a) Suppose that no string in S is a prefix of another string in S. Prove that $\sum_{k \geq 0} \frac{n_k}{2^k} \leq 1$.

   (b) [Bonus] Suppose that every binary string is obtained as the concatenation of strings in S in at most one way. (For example, we allow $S = \{0, 01\}$ but S cannot contain 0, 10, and 01 since $010 = 0(10)$ and $010 = (01)0$.) Prove that $\sum_{k \geq 0} \frac{n_k}{2^k} \leq 1$.

5. [CM] Let G be an n-vertex graph and let s be an integer with $1 \leq s \leq n$. Consider an experiment which an s-subset S of $V(G)$ is chosen uniformly at random. The subgraph of G induced by S is denoted $G[S]$.

   (a) Compute the expected number of components of $G[S]$ when G is a cycle and when G is a tree.

   (b) Compute the expected number of components of $G[S]$ having size k when G is a cycle and when G is a path.

6. A tournament G is k-egalitarian if, for each set of vertices S with $|S| \leq k$, there exists a vertex u such that $S \subseteq N^+(u)$. (Recall that $N^+(u)$ denotes the outneighborhood of u.)

   (a) Prove that if $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$, then there exists an n-vertex tournament that does not contain a dominating set of size k.

   (b) Let $f(k)$ be the minimum number of vertices in a k-egalitarian tournament. Conclude from (a) that $f(k) \leq (1 + o(1))(\ln 2)k^2 2^k$.