Directions: Solve 5 of the following 6 problems. All written work must be your own, and you must cite any external sources that you use.

1. [MR 2.4] Airlines find that each passenger who reserves a seat fails to turn up with probability 0.1 independently of the other passengers. So an airline sells 10 tickets for a 9 seat airplane and 20 tickets for an 18 seat airplane. Which type of plane is more frequently overbooked?

2. Let ε be a positive constant. Let X be the length of a longest run of heads in a sequence of m flips of a fair coin.

   (a) Prove that with high probability, $X \le (1+\varepsilon)\lg m$.

   (b) Prove that with high probability, $X \ge (1-\varepsilon)\lg m$.

   Comment: it can be shown that when c is a fixed integer, $\Pr(X = (\lg m) - 1 + c)$ approaches $e^{-2^{-(c+1)}} - e^{-2^{-c}}$ as m increases over powers of 2. The probability that X is exactly $(\lg m) - 1$ (the most likely value) approaches $e^{-1/2} - e^{-1} \approx 0.239$. The probability that $(\lg m) - 2 \le X \le \lg m$ tends to approximately 0.643. This is somewhat surprising. If you flip a coin 2^{20} times (over 1 million flips), the length of a longest run of heads will be either 18, 19, or 20 more than 64% time.

3. Let $x_1, \ldots, x_k$ be a list of integers, each of which is independently selected uniformly at random from $\{1, \ldots, n\}$.

   (a) Show that the probability that no repetitions occur is at most $e^{-\frac{1}{n}\binom{k}{2}}$.

   (b) Use part (a) to show that if $1 + \sqrt{n \ln n}$ objects are randomly selected with replacement from a set of size n, then with high probability some object is selected at least twice.

4. Recall that a *tournament* is an orientation of a complete graph. A *Hamiltonian path* is a path containing each vertex. Prove that for each n, there exists an n-vertex tournament with at least $n!/2^{n-1}$ Hamiltonian paths. For example, with $n = 3$, the directed 3-cycle has 3 Hamiltonian paths (one starting from each vertex).

5. Bipartite subgraphs.

   (a) Use the first moment method to show that every graph with m edges contains a bipartite subgraph with at least $m/2$ edges.

   (b) Modify your argument in part (a) to show that if G has m edges and a matching of size k, then G contains a bipartite subgraph with $(m+k)/2$ edges.

6. *A deterministic construction.* Let G_k be the graph whose vertices are the triples (x_1, x_2, x_3) such that $1 \le x_i \le k$ for each i with edges joining pairs that differ in exactly two coordinates. For example, in G_5 there is an edge joining $(3,4,5)$ and $(3,5,2)$, but $(3,4,5)$ and $(3,2,5)$ are nonadjacent.

   (a) Draw G_2, and use your drawing to describe G_2 in terms of familiar graphs.

   (b) By symmetry, all vertices in G_k have the same degree; that is, G_k is a *regular* graph. Find the degree of regularity.

   (c) Prove that $\omega(G_k) \le 3k - 2$. (Note: this can be improved to $\omega(G_k) \le \max\{4, k\}$ without much additional work.)

(d) Prove that $\alpha(G_k) \leq 3k$. Hint: if S is an independent set in G_k, form auxiliary graph H with vertex set S where x and y are adjacent if x and y differ in exactly one coordinate. For each edge $xy \in E(H)$, color xy with the coordinate in which x and y differ. So, H is a graph in which every edge receives one of three possible colors. What can you say about the structure of H?

(e) Conclude that if n is a perfect cube, then there exists an n-vertex graph G with $\max\{\alpha(G), \omega(G)\} \leq 3n^{1/3}$