Name: Solutions

Directions: Show all work. No credit for answers without work.

1. [**5 points**] Suppose that a motorboat is moving at 60 ft/s when its motor breaks, and that 20 seconds later, the boat moves at 30 ft/s. Assuming that the boat encounters resistance that is proportional to its velocity, how far does the boat coast in total?

(1) $\xrightarrow{v \text{ pos.}}$

- $Ma = F_{resistance}$
- $m\frac{dv}{dt} = -kv$
- $\frac{dv}{dt} = -\frac{k}{m}v$
- Set $\rho = \frac{k}{m}$, a positive const.
- $\frac{dv}{dt} = -\rho v$
- $\frac{1}{v}\frac{dv}{dt} = -\rho$
- $\int \frac{1}{v}\frac{dv}{\cancel{dt}} = \int -\rho \, dt$

$\ln|v| = -\rho t + C$

- $v > 0$:

$\ln(v) = -\rho t + C$

$v = C' e^{-\rho t}$

(2) $t=0$: $v=60$:

$60 = Ce^{-\rho 0}$

$C = 60$

- $v = 60e^{-\rho t}$

$t=20$, $v=30$:

$30 = 60e^{-\rho \cdot 20}$

$\frac{1}{2} = e^{-\rho 20}$

$-\ln(2) = -\rho \cdot 20$

$\rho = \frac{\ln(2)}{20}$

$v = 60e^{-\rho t} = 60e^{-\frac{\ln(2)}{20}t}$

$\frac{dx}{dt} = 60e^{-\frac{\ln(2)}{20}t}$

$x = 60\int e^{-\frac{\ln(2)}{20}t}\, dt$

$= -60\cdot\left(\frac{20}{\ln(2)}\right)e^{-\frac{\ln(2)}{20}t} + C$

$t=0$: $x=0$: $C = 60\left(\frac{20}{\ln(2)}\right)$

$x = 60\left(\frac{20}{\ln(2)}\right)\left[1 - e^{-\frac{\ln(2)}{20}t}\right]$

(3) As $t \to \infty$,

$x \to \boxed{60 \cdot \frac{20}{\ln(2)} \text{ ft}}$

$\approx \boxed{1731.23 \text{ ft}}$

2. **[3 points]** Given that $y_1 = 1$ and $y_2 = e^{3x}$ are linearly independent solutions to the homogeneous equation $y'' - 3y' = 0$, find a solution that also satisfies $y(0) = 3$ and $y'(0) = -2$.

$y = c_1 \cdot 1 + c_2 \cdot e^{3x}$

$y' = 0 + 3c_2 \cdot e^{3x}$

~~$y'' = 0 + 9c_2 e^{3x}$~~

$y(0)=3$: $\quad 3 = c_1 \cdot 1 + c_2 \cdot e^{3\cdot 0}$

$3 = c_1 + c_2$

$y'(0) = -2$: $\quad -2 = 0 + 3c_2 e^{3\cdot 0}$

$-2 = 3c_2$

$3 = c_1 + c_2$

$-2 = 3c_2$

$c_2 = -\frac{2}{3}$

$c_1 = 3 - c_2 = 3 + \frac{2}{3} = \frac{11}{3}$

$\boxed{y = \frac{11}{3} - \frac{2}{3} e^{3x}}$

3. **[2 points]** Complete the following definition. Functions $f_1, f_2, \ldots, f_t$ are linearly dependent on an open interval I if ...

there exist constants $c_1, c_2, \ldots, c_t$, not all zero, such that

$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_t f_t(x) = 0$

for all x in I.