Name: Key

**Directions:** This test has 6 pages, each worth 10 points. The test is scored out of 50 points. Your lowest scoring page is dropped. Unless explicitly stated, answers to counting problems do not need to be simplified.

1. [**5 points**] When $x$ is a string over $\{a, b\}$, let $x^D$ be the string obtained from $x$ by doubling each character in $x$. For example, if $x = abaa$, then $x^D = aabbaaaa$. Give a recursive definition for $x^D$.

$$x^D = \begin{cases} \lambda & \text{if } x = \lambda \\ aay^D & \text{if } x = ay \\ bby^D & \text{if } x = by \end{cases}$$

2. [**5 points**] Define a set of strings $S$ by the following rules.

   1. $\lambda \in S$ and $b \in S$
   2. if $x \in S$, then $ax \in S$
   3. if $x \in S$, then $bax \in S$

   Give an equivalent, non-recursive definition of $S$.

$S = \{\lambda, b, a, aa, ba, aaa, aba, baa, bab, \ldots\}$

$\boxed{S = \{x \mid \text{no two } b\text{'s appear next to each other in } x\}}$

3. Let $T(1) = -2$, $T(2) = 3$, and $T(n) = 2T(n-1) + 3T(n-2)$ for $n \geq 3$.

   (a) [**2 points**] Find $T(3)$ and $T(4)$.

$$T(3) = 2 \cdot 3 + 3 \cdot (-2) = 0$$
$$T(4) = 2 \cdot 0 + 3 \cdot 3 = 9$$

   (b) [**5 points**] Solve the recurrence relation.

1

$$x^2 = 2x + 3$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$
$$x = -1, 3$$

2

$$T(n) = p \cdot 3^{n-1} + q \cdot (-1)^{n-1}$$
$$-2 = p \cdot 3^0 + q \cdot (-1)^0$$
$$-2 = p + q \quad (1)$$
$$3 = p \cdot 3^1 + q(-1)^1$$
$$3 = 3p - q \quad (2)$$

3

Add (1) and (2):
$$1 = 4p$$
$$p = \frac{1}{4}$$

4

$$q = -2 - \frac{1}{4} = -\frac{9}{4}$$

5 Plug in p and q:

$$\boxed{T(n) = \frac{1}{4} \cdot 3^{n-1} - \frac{9}{4}(-1)^{n-1}}$$

4. [**3 points**] In the following algorithm, the work unit is the *write* statement, which produces some text on the screen. Determine exactly how many times the algorithm executes the *write* statement. You may assume that $n$ is even.

```
DoSomething(n):
  for k = 1 to n do
      t = n
      while t ≥ 2 do
          write t
          t = t − 2
      end while
  end for
```

For loop: $n$ times
while loop: $\frac{n}{2}$ times.

total $= n \cdot \frac{n}{2} = \boxed{\frac{n^2}{2} \text{ times}}$

5. **[6 parts, 1 point each]** Let $A = \{1, 2, \{3\}, \{4, 5\}, 3\}$, $B = \{2, 3, \{4\}\}$, and $C = \{1, \{2\}, \{3\}\}$.

(a) True or false: $2 \in B$.

(b) True or false: $4 \in B$.

(c) True or false: $B \subseteq A$.

(d) True or false: $\{3\} \subseteq C$.

(e) Find $A \cap C$.

$$A \cap C = \{1, \{3\}\}$$

(f) Find $\mathcal{P}(\emptyset)$.

$$\mathcal{P}(\emptyset) = \{\emptyset\}$$

6. **[4 points]** Let $S$ be the set of infinite sequences of zeros and ones. For example, both the sequence $000\ldots$ consisting of all zeros and the sequence $010101\ldots$ of alternating zeros and ones are members of $S$. Is $S$ denumerable/countable? Explain why your answer is correct.

S is not denumerable/countable. If S were countable, then we could list the elements of S in a table:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $s_1$: | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | ... |
| $s_2$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | ... |
| $s_3$ | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | ... |
| $s_4$ | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | ... |
| ⋮ | | | | | | | | | |
| $x$: | 1 | 1 | 0 | 0 | ... | | | | |

And then we could use Cantor's Diagonalization to construct a new string $x$ that is not on the list.

7. [**2 parts, 3 points each**] An ice cream store has 10 different flavors. John plans to visit the store each day for the next 4 days and buy an ice cream cone each time.

   (a) How many ways are there for John to buy ice cream?

$$10 \cdot 10 \cdot 10 \cdot 10 = \boxed{10^4} = \boxed{10{,}000}$$

   (b) John's favorite flavor is chocolate. How many ways can he buy ice cream if he wants to buy chocolate at least once?

Never buy chocolate: $9^4$

Buy chocolate $\geq 1$ time: $\boxed{10^4 - 9^4} = \boxed{3439}$

8. [**4 points**] Let $S$ be the set of ordered arrangements of $\{1, \ldots, 6\}$ in which at least one of the following happens: 1 is in the first position, 2 is in the second position, or 3 is in the third position. For example, 423165 and 165432 are in $S$, but 312456 is not. Find the size of $S$.

   *[Hint: Let $A_1$ be the set of arrangements where 1 is in the first position, let $A_2$ be the set of arrangements where 2 is in the second position, and let $A_3$ be the set of arrangements where 3 is in the third position. How are $S$ and $A_1, A_2, A_3$ related?]*

- $S = A_1 \cup A_2 \cup A_3$; Use Inclusion/Exclusion.
- $|A_1| = |A_2| = |A_3| = 5!$
- $|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = 4!$
- $|A_1 \cap A_2 \cap A_3| = 3!$
- $|S| = |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

4

$$= \boxed{3 \cdot 5! - 3 \cdot 4! + 3!} = \boxed{294}$$

9. **[3 points]** Find the numerical value of $C(7,3)$.

$$C(7,3) = \frac{7!}{3! \cdot (7-3)!} = \frac{7!}{3! \cdot 4!} = \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{3! \cdot \cancel{4!}} = \frac{7 \cdot 6 \cdot 5}{\cancel{3 \cdot 2}} = \boxed{35}$$

10. A group of 8 men and 10 women gather to form a committee.

(a) **[3 points]** How many ways are there to form a committee of 6 people from the group?

18 people in total

$$\boxed{C(18,6) = 18,564}$$

(b) **[4 points]** How many ways are there to form a committee of 6 people, if the committee must have at least one man and at least one woman?

Total ways: $C(18,6)$

no men: $C(10,6)$

no women: $C(8,6)$

at least one man and at least one woman : $\boxed{C(18,6) - C(10,6) - C(8,6)}$

11. [**3 points**] An unbalanced 6-sided die has the following probability distribution:

| $x_i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $p(x_i)$ | 1/12 | 3/12 | 3/12 | 2/12 | 1/12 | 2/12 |

Find the probability that an even number is rolled.

$$P(\text{even}) = \frac{3}{12} + \frac{2}{12} + \frac{2}{12} = \boxed{\frac{7}{12}}$$

12. A family has 4 children; each child is as likely to be a boy as a girl.

(a) [**3 points**] What is the probability that at least 3 of the children are boys?

- $|S| = 2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$
- $E = \{ BBBB, GBBB, BGBB, BBGB, BBBG \}$

$$P(E) = \frac{|E|}{|S|} = \boxed{\frac{5}{16}}$$

(b) [**4 points**] What is the probability that the first child is a boy given that at least 3 of the children are boys?

$E_1$ = at least 3 children are boys

$E_2$ = first child is a boy

$E_1 \cap E_2 = \{ BBBB, BGBB, BBGB, BBBG \}$

$$P(E_2 | E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)} = \frac{\frac{4}{16}}{\frac{5}{16}} = \boxed{\frac{4}{5}}$$