Name: Key

1. **[2 parts, 1 point each]** A collection S of strings is defined recursively by

   1. The empty string λ belongs to S.
   2. If X belongs to S, then bX and Xa belong to S.

   (a) Write down all the strings of length 4 that are in S.

   bbbb, bbba, bbaa, baaa, aaad

   (b) Give a simple, non-recursive definition of S that is equivalent to the given definition.

   S is the set of strings with characters a and b where every b comes before every a.

2. **[4 parts, 2 points each]** Solve the following recurrence relations.

   (a) $T(n) = \begin{cases} 1 & n = 1 \\ T(n-1) + 2 & n \geq 2 \end{cases}$

$$T(n) = T(n-1) + 2$$
$$= (T(n-2) + 2) + 2$$
$$\vdots$$
$$= T(1) + \underbrace{2 + 2 + \cdots + 2}_{n-1 \text{ terms}} = 1 + 2(n-1) = \boxed{2n - 1}$$

   (b) $T(n) = \begin{cases} 1 & n = 1 \\ 3nT(n-1) & n \geq 2 \end{cases}$

$$T(n) = 3n\,T(n-1)$$
$$= 3n\big(3(n-1)T(n-2)\big) = 3 \cdot 3 \cdot n(n-1) \cdot T(n-2)$$
$$= \cancel{3n(3(n-1)} \; 3 \cdot 3 \cdot n(n-1) \cdot \left(3 \overset{(n-2)}{\cdot} T(n-3)\right) = 3^3\, n(n-1)(n-2) \cdot T(n-3)$$
$$= \boxed{3^{n-1} \cdot n!}$$

(c) $T(n) = \begin{cases} 1 & n = 1 \\ -3 & n = 2 \\ 5T(n-1) - 6T(n-2) & n \geq 3 \end{cases}$

$t^2 = 5t - 6$

$t^2 - 5t + 6 = 0$

$(t-3)(t-2) = 0$

$r_1 = 3, \; r_2 = 2$

$T(n) = p3^{n-1} + q2^{n-1}$

$T(1): \; 1 = p + q$

$T(2): \; -3 = p3 + q \cdot 2$

$-3 = -3p - 3q$

$-3 = 3p + 2q$

$-6 = -q$

$q = 6$

$p = 1 - q = 1 - 6 = 5$

$\boxed{T(n) = (-5) \cdot 3^{n-1} + 6 \cdot 2^{n-1}}$

(d) $T(n) = \begin{cases} 2 & n = 1 \\ 1 & n = 2 \\ -2T(n-1) - T(n-2) & n \geq 3 \end{cases}$

$t^2 = -2t - \cancel{t}$

$t^2 + 2t + 1 = 0$

$(t+1)(t+1) = 0$

· One root: $r = -1$.

$T(n) = p(-1)^{n-1} + q(n-1)(-1)^{n-1}$

· $T(1)$: $2 = p$

· $T(2)$: $1 = p(-1) + q(-1)$

$1 = -p - q$

$1 + p = -q$

$3 = -q$

$q = -3$

$\boxed{T(n) = 2(-1)^{n-1} - 3(n-1)(-1)^{n-1}}$