Name: _____

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[4 parts, 4 points each]** Identify the logical connectors and give their definitions.

   (a) $P \vee Q$

   (b) $P \Leftrightarrow Q$

   (c) $\forall x \in S, P(x)$

   (d) $\sim P$

2. **[4 points]** What is the difference between a Lemma and a Theorem?

3. **[4 points]** Give the definitions of even and odd integers.

4. **[2 parts, 5 points each]** Let φ be the statement $(P \vee Q) \wedge (Q \Rightarrow P)$.

   (a) Give the truth table for φ.

   (b) Give a simple sentence that is logically equivalent to φ.

5. **[3 parts, 5 points each]** Translate the following statements in symbolic logic into English as naturally as possible. Indicate if the statement is true or false by writing the entire word.

$$P(a,b) : \exists c \in \mathbb{Z}, a < c < b \qquad\qquad Q(a,b) : ab \geq 0$$

   (a) $P(\sqrt{2}, \pi)$

   (b) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x - y| \geq 5 \Rightarrow Q(x,y)$

   (c) $\forall S \subseteq \mathbb{R}, |S| \geq 3 \implies \exists x \in S, \exists y \in S, x \neq y \wedge Q(x,y)$

6. **[2 parts, 5 points each]** Negate each sentence below in English as naturally as possible.

   (a) Every integer gets larger when multiplied by two.

   (b) For all positive real numbers x and y, if $x < 1$, then there exists an integer z such that $y < xz < y + 1$.

7. **[2 parts, 5 points each]** Let $\phi(x, y, z)$ be the implication "If $x < y$ and $y < z$, then $x < z$." Write the following implications in English.

   (a) The converse of $\phi(x, y, z)$.

   (b) The contrapositive of $\phi(x, y, z)$.

8. **[7 points]** Let $x \in \mathbb{Z}$. Prove that if x is even, then $x^2 + 1$ is odd.

9. **[8 points]** Let $x, y \in \mathbb{R}$. Prove that if $x \neq y$, then $(x+y)^2 > 4xy$.

10. **[8 points]** Note that $x \nmid y$ means that x does not divide y. Let $a, b, d \in \mathbb{Z}$. Prove that if $d^2 \nmid ab$, then $d \nmid a$ or $d \nmid b$.

11. **[8 points]** Let $a \in \mathbb{Z}$. Without using prime factorizations, prove that if $2 \mid a$ and $3 \mid a$, then $6 \mid a$. Hint: apply the division algorithm.

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