Name: _____

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[4 parts, 4 points each]** Identify the logical connectors and give their definitions.

   (a) $P \vee Q$

   Solution: Logical Or. This is a statement that asserts that at least one of P or Q is true. ■

   (b) $P \Leftrightarrow Q$

   Solution: Biconditional. This is a statement that asserts that P is true if and only if Q is true; equivalently, it asserts that P and Q have the same truth value. ■

   (c) $\forall x \in S, P(x)$

   Solution: Universal quantifier. This is a statement that asserts that $P(x)$ holds for all $x \in S$. ■

   (d) $\sim P$

   Solution: Logical negation. This is a statement that asserts that P is false. ■

2. **[4 points]** What is the difference between a Lemma and a Theorem?

   Solution: A Lemma is a statement with a known proof that may not be of independent interest but is used to prove other results. A Theorem is a statement with a known proof that is significant and independently interesting. ■

3. **[4 points]** Give the definitions of even and odd integers.

   Solution: An integer n is *even* if $n = 2k$ for some $k \in \mathbb{Z}$. An integer n is *odd* if $n = 2k + 1$ for some $k \in \mathbb{Z}$. ■

4. **[2 parts, 5 points each]** Let φ be the statement $(P \vee Q) \wedge (Q \Rightarrow P)$.

   (a) Give the truth table for φ.

   Solution:

P	Q	$P \vee Q$	$Q \Rightarrow P$	$(P \vee Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T
T	F	T	T	T
F	T	T	F	F
F	F	F	T	F

   ■

   (b) Give a simple sentence that is logically equivalent to φ.

   Solution: From the truth table, we see that φ is equivalent to P. ■

5. **[3 parts, 5 points each]** Translate the following statements in symbolic logic into English as naturally as possible. Indicate if the statement is true or false by writing the entire word.

$$P(a,b) : \exists c \in \mathbb{Z}, a < c < b \qquad\qquad Q(a,b) : ab \geq 0$$

   (a) $P(\sqrt{2}, \pi)$

   Solution: There is an integer strictly between $\sqrt{2}$ and π.
   This is **true**; we have $\sqrt{2} < 2 < \pi$. ■

   (b) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x - y| \geq 5 \Rightarrow Q(x, y)$

   Solution: If two real numbers are are at least at distance 5, then their product is at least zero.
   This is **false**. For example, if $x = -10$ and $y = 10$, then $|x - y| = 20$ but $xy < 0$. ■

   (c) $\forall S \subseteq \mathbb{R}, |S| \geq 3 \implies \exists x \in S, \exists y \in S, x \neq y \wedge Q(x, y)$

   Solution: Every set of real numbers of size at least 3 contains a pair of distinct real numbers whose product is at least zero.
   This is **true**. If $0 \in S$, then we may take $x = 0$ and y to be any other element of S. If $0 \notin S$, then since $|S| \geq 3$, either S has two positive real numbers or two negative real numbers. Taking x and y to be distinct real numbers in S with the same sign, we have $xy > 0$. ■

6. **[2 parts, 5 points each]** Negate each sentence below in English as naturally as possible.

   (a) Every integer gets larger when multiplied by two.

   Solution: There exists an integer k such that $2k \leq k$. ■

   (b) For all positive real numbers x and y, if $x < 1$, then there exists an integer z such that $y < xz < y + 1$.

   Solution: There exist positive real numbers x and y such that $x < 1$ and every integer z satisfies $y \geq xz$ or $xz \geq y + 1$. ■

7. **[2 parts, 5 points each]** Let $\phi(x, y, z)$ be the implication "If $x < y$ and $y < z$, then $x < z$." Write the following implications in English.

   (a) The converse of $\phi(x, y, z)$.

   Solution: If $x < z$, then $x < y$ and $y < z$. ■

   (b) The contrapositive of $\phi(x, y, z)$.

   Solution: If $x \geq z$, then $x \geq y$ or $y \geq z$. ■

8. **[7 points]** Let $x \in \mathbb{Z}$. Prove that if x is even, then $x^2 + 1$ is odd.

   Solution: Since x is even, we have that $x = 2k$ for some $k \in \mathbb{Z}$. We compute

$$\begin{aligned} x^2 + 1 &= (2k)^2 + 1 \\ &= 2(2k) + 1. \end{aligned}$$

   Since $2k \in \mathbb{Z}$, it follows that $2(2k) + 1$ is odd and hence $x^2 + 1$ is odd. ■

9. **[8 points]** Let $x, y \in \mathbb{R}$. Prove that if $x \neq y$, then $(x+y)^2 > 4xy$.

   Solution: Let x and y be distinct real numbers. Since $x \neq y$, we have that $(x-y)^2 > 0$, or $x^2 - 2xy + y^2 > 0$. Adding $4xy$ to both sides gives $x^2 + 2xy + y^2 > 4xy$, and so $(x+y)^2 = x^2 + 2xy + y^2 > 4xy$. ■

10. **[8 points]** Note that $x \nmid y$ means that x does not divide y. Let $a, b, d \in \mathbb{Z}$. Prove that if $d^2 \nmid ab$, then $d \nmid a$ or $d \nmid b$.

    Solution: We prove the contrapositive: if $d \mid a$ and $d \mid b$, then $d^2 \mid ab$. Since $d \mid a$ and $d \mid b$, we have that $a = k_1 d$ and $b = k_2 d$ for some $k_1, k_2 \in \mathbb{Z}$. It follows that $ab = (k_1 d)(k_2 d) = (k_1 k_2)d^2$. Since $k_1 k_2 \in \mathbb{Z}$, it follows that $d^2 \mid ab$. ■

11. **[8 points]** Let $a \in \mathbb{Z}$. Without using prime factorizations, prove that if $2 \mid a$ and $3 \mid a$, then $6 \mid a$. Hint: apply the division algorithm.

    Solution: Suppose that $2 \mid a$ and $3 \mid a$. It follows that $a = 2k_1$ and $a = 3k_2$ for some $k_1, k_2 \in \mathbb{Z}$. By the division algorithm, we have that $a = 6q + r$ for some $q, r \in \mathbb{Z}$ with $0 \leq r < 6$. Solving for r and substituting for a gives $r = a - 6q = 2k_1 - 6q = 2(k_1 - 3q)$ and $r = a - 6q = 3k_2 - 6q = 3(k_2 - 2q)$. Since $k_1 - 3q$ and $k_2 - 2q$ are both integers, it follows that $2 \mid r$ and $3 \mid r$.

    Since $r \in \mathbb{Z}$ and $0 \leq r < 6$, it follows that $r \in \{0, 1, 2, 3, 4, 5\}$. Since $2 \mid r$, it must be that $r \in \{0, 2, 4\}$. Since $3 \mid r$, it must be that $r \in \{0, 3\}$. The only possible value for r that is divisible by both 2 and 3 is 0, and so $r = 0$. Hence $a = 6q$ and so $6 \mid a$. ■