Name: _____

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. Use any proof technique.

   (a) **[3 points]** An integer a is even if and only if a^2 is even.

   Solution: Let $a \in \mathbb{Z}$. To prove a biconditional, we prove the forward and backward implications.

   ($\Longrightarrow$): If a is even, then a^2 is even. Suppose a is even. We have that $a = 2k$ for some $k \in \mathbb{Z}$, and we compute $a^2 = (2k)^2 = 2(2k^2)$. Since $2k^2 \in \mathbb{Z}$, it follows that a^2 is even.

   ($\Longleftarrow$): If a^2 is even, then a is even. We prove the contrapositive: if a is odd, then a^2 is odd. Suppose that a is odd. We have that $a = 2k + 1$ for some $k \in \mathbb{Z}$, and we compute $a^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since $2k^2 + 2k \in \mathbb{Z}$, it follows that a^2 is odd. ■

   (b) **[2 points]** Let d and n be integers with $d \geq 2$. Show that $d \nmid n$ or $d \nmid 2n + 1$.

   Solution: Let d and n be integers with $d \geq 2$, and suppose for a contradiction that $d \mid n$ and $d \mid 2n + 1$. We have that $n = dk$ and $2n + 1 = d\ell$ for some $k, \ell \in \mathbb{Z}$. Substituting our expression for n into the second equation gives $2(dk) + 1 = d\ell$, or $1 = d\ell - 2(dk) = d(\ell - 2k)$. Since $\ell - 2k \in \mathbb{Z}$ we have that $d \mid 1$. But this contradicts that $d \geq 2$. The contradiction implies that either $d \nmid n$ or $d \nmid 2n + 1$. ■

2. Irrational Numbers

(a) **[3 points]** Prove that $\sqrt{6}$ is irrational.

Solution: Suppose for a contradiction that $\sqrt{6}$ is irrational. We may write $\sqrt{6} = \frac{a}{b}$ such that $a, b \in \mathbb{Z}$ and a and b have no common divisors except 1 and -1. Squaring both sides and rearranging, it follows that $6b^2 = a^2$, or $2(3b^2) = a^2$. It follows that a^2 is even. By Question 1(a), it follows that a is even, and so $a = 2k$ for some $k \in \mathbb{Z}$. Hence $6b^2 = a^2 = (2k)^2 = 4k^2$, and so $3b^2 = 2k^2$. It follows that $3b^2$ is even. Since 3 is odd, and an odd number times an odd number is an odd number, it must be that b^2 is even. Hence b is even by Question 1(a). Therefore both a and b are both divisible by 2, contradicting that a and b have no common divisors except 1 and -1. ■

(b) **[2 points]** Prove that $\sqrt{2} + \sqrt{3}$ is irrational. (Hint: suppose for a contradiction that this number is rational. What can we say about $(\sqrt{2} + \sqrt{3})^2$?)

Solution: Suppose for a contradiction that $\sqrt{2} + \sqrt{3} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$. Squaring both sides gives $\frac{a^2}{b^2} = (\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{2}\sqrt{3} + 3 = 5 + \sqrt{6}$. It follows that $\sqrt{6} = \frac{a^2}{b^2} - 5 = \frac{a^2 - 5b^2}{b^2}$. Since $a^2 - 5b^2$ and b^2 are integers, we have that $\sqrt{6}$ is rational. But this contradicts part (a). ■