Name: _____

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[2 points]** Let $x, y \in \mathbb{Z}$. Give a proof by contrapositive that if xy is even, then x is even or y is even.

   Solution: We prove the contrapositive: if x and y are odd, then xy is odd. Let x and y be odd integers. By definition, we have that $x = 2k_1 + 1$ and $y = 2k_2 + 1$ for some $k_1, k_2 \in \mathbb{Z}$. We compute $xy = (2k_1+1)(2k_2+1) = 4k_1k_2 + 2k_1 + 2k_2 + 1 = 2(2k_1k_2 + k_1 + k_2) + 1$. Since $2k_1k_2 + k_1 + k_2 \in \mathbb{Z}$, it follows that xy is odd. ■

2. **[2 parts, 2 points each]** Congruences.

   (a) Give the definition of $a \equiv b \pmod{m}$.

   Solution: The integers a and b are *congruent modulo* m, denoted $a \equiv b \pmod{m}$, if $m \mid a - b$. ■

   (b) Let $a, b, m \in \mathbb{Z}$ and let m be odd. Show that if $2a \equiv 2b \pmod{m}$, then $a \equiv b \pmod{m}$.

   Solution: Since $2a \equiv 2b \pmod{m}$, we have that $m \mid 2a-2b$, and so $2(a-b) = 2a-2b = mk$ for some $k \in \mathbb{Z}$. Since $mk = 2(a-b)$ and $a - b \in \mathbb{Z}$, we have that mk is even. By Question 1, we have that m is even or k is even. But we know that m is odd, and so it must be that k is even. Since k is even, we have that $k = 2\ell$ for some $\ell \in \mathbb{Z}$. Therefore $2(a-b) = mk = m(2\ell)$ and dividing by two gives $a - b = m\ell$. Since $\ell \in \mathbb{Z}$ it follows that $m \mid a - b$ and hence by definition $a \equiv b \pmod{m}$. ■

3. **[2 points]** Let $a \in \mathbb{Z}$. Prove that if $9 \nmid a^2$, then $3 \nmid a$.

   Solution: We prove the contrapositive: if $3 \mid a$, then $9 \mid a^2$. Suppose that $3 \mid a$. By definition, we have that $a = 3k$ for some $k \in \mathbb{Z}$. We compute $a^2 = (3k)^2 = 9k^2$. Since $k^2 \in \mathbb{Z}$, it follows that $9 \mid a^2$. ■

4. **[2 points]** Prove that for each $n \in \mathbb{N}$, there exist nonnegative integers r and s such that s is odd and $n = 2^r s$.

   Solution: Let r be the maximum integer such that $2^r \mid n$. Note that $2^0 = 1$ and $1 \mid n$, and therefore $r \geq 0$. Since $2^r \mid n$, we have that $n = 2^r s$ for some $s \in \mathbb{Z}$. Since n and 2^r are both positive integers, it follows that s is also positive. It remains to show that s is odd.

   Suppose for a contradiction that s is even, meaning that $s = 2k$ for some $k \in \mathbb{Z}$. We have that $n = 2^r s = 2^r(2k) = 2^{r+1}k$. Since $k \in \mathbb{Z}$, it follows that $2^{r+1} \mid n$. But this contradicts that r is the maximum integer such that $2^r \mid n$. The contradiction implies that s is odd.

   Hence we have that $n = 2^r s$, where r and s are nonnegative integers and s is odd. ■