Name: _____

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[3 points]** Recall that an integer p is *prime* if $p \geq 2$ and the only positive divisors of p are 1 and p. Let $a, b \in \mathbb{Z}$. Prove that $\gcd(a, b) = 1$ if and only if for each prime p, either $p \nmid a$ or $p \nmid b$. (You may use that each $n \in \mathbb{Z}$ with $n \geq 2$ is divisible by some prime.)

   Solution: ($\Longrightarrow$) We prove the contrapositive: if there exists a prime p such that $p \mid a$ and $p \mid b$, then $\gcd(a, b) \neq 1$. Indeed, if $p \mid a$ and $p \mid b$, then p is a common divisor of a and b and therefore $\gcd(a, b) \geq p \geq 2$, and so $\gcd(a, b) \neq 1$.

   ($\Longleftarrow$) We again prove the contrapositive: if $\gcd(a, b) \neq 1$, then there exists a prime p such that $p \mid a$ and $p \mid b$. Suppose that $\gcd(a, b) \neq 1$. Since always $\gcd(a, b) \geq 1$, we have that $\gcd(a, b) \geq 2$. Let $d = \gcd(a, b)$; since $d \geq 2$, we there is a prime p such that $p \mid d$. Since $p \mid d$ and $d \mid a$, we have that $p \mid a$. Similarly, $p \mid b$. Hence there exists a prime that is a common divisor of a and b. ■

2. Existence and Uniqueness.

   (a) **[3 points]** Prove that there is a unique integer d such that $d > 0$ and d divides every integer.

   Solution: (Existence) Let $d = 1$. Clearly, $d > 0$. Also, if $n \in \mathbb{Z}$, we have that $n = 1 \cdot n = dn$ and it follows that $d \mid n$.

   (Uniqueness) Suppose that d is a positive integer and d divides every integer. In particular, $d \mid 1$, and so $1 = dk$ for some $k \in \mathbb{Z}$. It follows from $1 = dk$ and $d, k \in \mathbb{Z}$ that either $d = k = 1$ or $d = k = -1$. Since $d > 0$, we have $d = k = 1$. Hence the only positive integer that divides every integer is 1. ■

   (b) **[1 point]** What goes wrong if the condition $d > 0$ is omitted from the statement to be proved in part (a)?

   Solution: Uniqueness fails, since both 1 and -1 divide every integer. ■

3. **[3 points]** Let A, B, C be sets. Let $X = A \cup (B \cap C)$ and let $Y = (A \cup B) \cap (A \cup C)$. Prove that $X = Y$.

   Solution: $(X \subseteq Y)$. Let $x \in X$. By definition of set union, we have that $x \in A$ or $x \in B \cap C$. We consider these cases separately.

   First, suppose that $x \in A$. We have that $x \in A \cup B$ and $x \in A \cup C$, and so $x \in (A \cup B) \cap (A \cup C) = Y$.

   Next, suppose that $x \in B \cap C$. Then $x \in B$ and $x \in C$. Since $x \in B$, we have that $x \in A \cup B$. Similarly, since $x \in C$, we have that $x \in A \cup C$. Therefore $x \in Y$.

   In both cases $x \in Y$. Since every element in X is also an element in Y, we have $X \subseteq Y$.

   $(Y \subseteq X)$. Let $y \in Y$. By definition of set intersection, we have that $y \in A \cup B$ and $y \in A \cup C$. Again, we consider two cases.

   First, suppose that $y \in A$. It then follows that $y \in A \cup (B \cap C) = X$ by definition of set union.

   Next, suppose that $y \notin A$. Since $y \in A \cup B$ but $y \notin A$, it must be that $y \in B$. Similarly, since $y \in A \cup C$ but $y \notin A$, we have that $y \in C$. Since $y \in B$ and $y \in C$, it follows that $y \in B \cap C$. Hence $y \in X$.

   In both cases $y \in X$ and so $Y \subseteq X$. ■