Solutions

**Directions:**

1. Write your name with one character in each box below.
2. Show all work. No credit for answers without work.
3. You are permitted to use one 8.5 inch by 11 inch sheet of prepared notes. No other aides are allowed.

1. **[15 points]** Determine whether the following vectors are linearly independent. If the vectors are linearly dependent, give a dependence relation.

$$\begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ -9 \\ -6 \\ 9 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 & -3 & 0 \\ -1 & -9 & 0 & 4 \\ 2 & -6 & 5 & -1 \\ 3 & 9 & -1 & -2 \end{bmatrix} \xrightarrow[\substack{R3 \overset{+}{=} -2R1 \\ R4 \overset{+}{=} -3R1}]{R2 \overset{+}{=} R1} \begin{bmatrix} 1 & 7 & -3 & 0 \\ 0 & -2 & -3 & 4 \\ 0 & -20 & 11 & -1 \\ 0 & -12 & 8 & -2 \end{bmatrix} \xrightarrow[R4 \overset{+}{=} (-6)R2]{R3 \overset{+}{=} (-10)R2} \begin{bmatrix} 1 & 7 & -3 & 0 \\ 0 & -2 & -3 & 4 \\ 0 & 0 & 41 & -41 \\ 0 & 0 & 26 & -26 \end{bmatrix} \xrightarrow[R4 \doteq \frac{1}{26}]{R3 \doteq \frac{1}{41}} \begin{bmatrix} 1 & 7 & -3 & 0 \\ 0 & -2 & -3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R4 \overset{+}{=} -R3} \begin{bmatrix} 1 & 7 & -3 & 0 \\ 0 & -2 & -3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R2 \overset{+}{=} 3R3]{R1 \overset{+}{=} 3R3} \begin{bmatrix} 1 & 7 & 0 & -3 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \doteq -\frac{1}{2}} \begin{bmatrix} 1 & 7 & 0 & -3 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \overset{+}{=} -7R2} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{No pivot, lin. dependent}}$

Take $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$:

$$\boxed{(-1)\begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} + (1)\begin{bmatrix} 7 \\ -9 \\ -6 \\ 9 \end{bmatrix} + (2)\begin{bmatrix} -3 \\ 0 \\ 5 \\ -1 \end{bmatrix} + (2)\begin{bmatrix} 0 \\ 4 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

2. **[10 points]** Characterize when the following vectors are linearly dependent in terms of simple conditions on $h$ and $k$.

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ h \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ k \\ -h \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 1 \\ k \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 3 & 1 \\ 0 & 6 & k & 5 \\ 0 & h & -h & 1 \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If $h=0$, then the $3^{rd}$ column has no pivot and the vectors are lin. dependent. Suppose $h \neq 0$.

$$R3 \overset{+}{=} -\frac{h}{6}(R2) \rightsquigarrow \begin{bmatrix} 2 & 4 & 3 & 1 \\ 0 & 6 & k & 5 \\ 0 & 0 & -h-\frac{hk}{6} & 1-\frac{5h}{6} \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If $-h - \frac{hk}{6} = 0$, then the $3^{rd}$ column has no pivot.

Simplify: $\cancel{-h}\left(1 + \frac{k}{6}\right) = 0 \iff \frac{k}{6} = -1 \iff k = -6$.

Suppose $k \neq -6$.

The $4^{th}$ column has no pivot if $k=0$.

So the vectors are lin. dependent if and only if $\boxed{h=0 \text{ or } k=-6 \text{ or } k=0.}$

3. Transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.

 (a) **[9 points]** Let $T_1$ be the transform given by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ x_2+1 \end{bmatrix}$. Is $T_1$ one-to-one/injective? Is $T_1$ onto/surjective? Is $T_1$ linear? Explain.

injective:

$\boxed{\text{No.}}$ $T_1\left(\begin{bmatrix} 5 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $T_1\left(\begin{bmatrix} 6 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so $T_1$ is not injective.

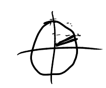
onto:

$\boxed{\text{No.}}$ No point in $\mathbb{R}^2$ gets mapped to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Linear?

$\boxed{\text{No.}}$ Since $T_1$ maps $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and not the zero vector, $T_1$ cannot be linear.

 (b) **[6 points]** Let $T_2$ be the transform given by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1+x_2 \\ x_2 \end{bmatrix}$ and let $T_3$ be the transformation that rotates points by $\pi/6$ radians. Find the standard matrix for the composition transformation $T_2 \circ T_3$ given by $\mathbf{x} \mapsto T_2(T_3(\mathbf{x}))$.



$$T_2(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad T_3(\vec{x}) = \begin{bmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

$$T_2(T_3(\vec{x})) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3}+1 & \sqrt{3}-1 \\ 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So standard matrix is $\boxed{\begin{bmatrix} \frac{\sqrt{3}+1}{2} & \frac{\sqrt{3}-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}}$

4. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transform, and let $A$ be the standard matrix for $T$.

 (a) **[2 points]** How many rows does $A$ have? How many columns?

A has $m$ rows and $n$ columns.

 (b) **[8 points]** By analyzing the pivot positions of $A$, prove that if $n > m$ then $T$ is not one-to-one/injective.

Since A has more columns than rows and every row has at most 1 pivot position, there must be a column of A that has no pivot position. This column represents a free variable in the homogeneous system $A\vec{x}=\vec{0}$, and so $A\vec{x}=\vec{0}_m$ for infinitely many vectors $\vec{x} \in \mathbb{R}^n$. Therefore $T$ is not injective.

5. **[20 points]** Find the inverse of the following matrix.

$$\begin{bmatrix} -8 & 1 & -9 \\ 3 & 0 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} -8 & 1 & -9 & 1 & 0 & 0 \\ 3 & 0 & 4 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R1 \leftrightarrow R3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 3 & 0 & 4 & 0 & 1 & 0 \\ -8 & 1 & -9 & 1 & 0 & 0 \end{array}\right] \xrightarrow[R3 \mathrel{+}= 8R1]{R2 \mathrel{+}= (-3)R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -3 \\ 0 & 1 & -1 & 1 & 0 & 8 \end{array}\right]$$

$$\xrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 8 \\ 0 & 0 & 1 & 0 & 1 & -3 \end{array}\right] \xrightarrow[R2 \mathrel{+}= R3]{R1 \mathrel{+}= (-1)R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 0 & 1 & -3 \end{array}\right]$$

So the inverse is $\boxed{\begin{bmatrix} 0 & -1 & 4 \\ 1 & 1 & 5 \\ 0 & 1 & -3 \end{bmatrix}}$.

6. **[10 points]** Find elementary matrices $E_1, E_2, E_3$ such that $E_3 E_2 E_1 A = B$.

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 1 & 2 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

A:

$$\begin{bmatrix} 1 & 2 & 7 \\ 1 & 2 & 8 \end{bmatrix} \overset{\substack{E_1 \\ R2 \mathrel{+}= (-1)R1}}{\sim} \begin{bmatrix} 1 & 2 & 7 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_2]{R1 \mathrel{+}= (-7)R2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \end{bmatrix} \overset{R2 \mathrel{+}= (-2)R1}{\sim} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R2 \mathrel{+}= 2R1$

$E_3$

$$\boxed{\underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{E_3} \underbrace{\begin{bmatrix} 1 & -7 \\ 0 & 1 \end{bmatrix}}_{E_2} \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}_{E_1} A = B.}$$

7. **[10 parts, 2 points each]** True/False. Assume the matrix operations below are well-defined. Justify your answers.

(a) Every elementary matrix is square.

$\boxed{\text{True.}}$ Every elementary matrix is invertible and so must be square.

(b) $(A+B)C = AC + BC$

$\boxed{\text{True}}$ This is the distributive property.

(c) $(A+B)(A+B) = A^2 + 2AB + B^2$

$\boxed{\text{FALSE}}$ $(A+B)(A+B) = A(A+B) + B(A+B) = A^2 + AB + BA + B^2$, and this equals $A^2 + 2AB + B^2$ only if $AB = BA$.

(d) If $AB = BA$, then $A$ and $B$ are inverses of one another.

$\boxed{\text{FALSE}}$. If $A = B$, then $AB = BA$ but typically a matrix is not its own inverse.

(e) If $A$ and $B$ are invertible $(n \times n)$-matrices, then $A$ and $B$ are row-equivalent.

$\boxed{\text{True}}$ By the I.M.T, both $A$ and $B$ are row-equivalent to $I_n$. It follows that they are row-equivalent to each other.

(f) If $A$ and $B$ are invertible $(n \times n)$-matrices, then $A + B$ is also invertible.

$\boxed{\text{FALSE}}$ Let $A = I_n$, $B = -I_n$. Now $A$ and $B$ are invertible but $A+B$ is the zero matrix, which is clearly not invertible.

(g) If $A$ and $B$ are invertible $(n \times n)$-matrices, then $AB$ is also invertible.

$\boxed{\text{TRUE}}$ $(AB)^{-1} = B^{-1}A^{-1}$.

(h) An $(n \times n)$-matrix $A$ is invertible if and only if its transpose $A^T$ is invertible.

$\boxed{\text{True}}$ By IMT.

(i) If $S$ and $T$ are linear transforms from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $S$ and $T$ are equal on at least $n$ points in $\mathbb{R}^n$, then $S = T$.

$\boxed{\text{FALSE}}$ This is true if $S$ and $T$ are equal on $n$ points that span $\mathbb{R}^n$ but false in general. For example $S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$.

(j) If $T$ is a linear transform and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is also linearly independent.

$\boxed{\text{FALSE}}$. If $T$ is the zero transform $T(\vec{x}) = \vec{0}$, and $\vec{v}_1 \neq \vec{0}$, then $\{\vec{v}_1\}$ is linearly independent but $\{T(\vec{v})\}$ is linearly dependent.



The converse is true: if $\{T(\vec{v}_1), \ldots, T(\vec{v}_k)\}$ is lin. independent, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is lin. independent.