Math343 **Quiz 9** **Oct 24, 2025**

Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[3 points]** Determine if the following vectors form a basis for $\mathbb{R}^4$.

$$\overset{v_1}{\begin{bmatrix} -3 \\ 1 \\ 0 \\ -2 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 6 \\ -4 \\ -1 \\ 6 \end{bmatrix}}, \overset{v_3}{\begin{bmatrix} 1 \\ -6 \\ -3 \\ 6 \end{bmatrix}}, \overset{v_4}{\begin{bmatrix} -8 \\ 1 \\ 0 \\ -2 \end{bmatrix}}$$

$$\begin{bmatrix} -3 & 6 & 1 & -8 \\ 1 & -4 & -6 & 1 \\ 0 & -1 & -3 & 0 \\ -2 & 6 & 6 & -2 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & -4 & -6 & 1 \\ -3 & 6 & 1 & -8 \\ 0 & -1 & -3 & 0 \\ -2 & 6 & 6 & -2 \end{bmatrix} \xrightarrow[R4 \mathrel{+}= 2R1]{R2 \mathrel{+}= 3R1} \begin{bmatrix} 1 & -4 & -6 & 1 \\ 0 & -6 & -17 & -5 \\ 0 & -1 & -3 & 0 \\ 0 & -2 & -6 & 0 \end{bmatrix} \xrightarrow[R3 \leftrightarrow R2]{R3 \mathrel{\div}= -1}$$

$$\begin{bmatrix} 1 & -4 & -6 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & -6 & -17 & -5 \\ 0 & -2 & -6 & 0 \end{bmatrix} \xrightarrow[R4 \mathrel{+}= 2R2]{R3 \mathrel{+}= 6R2} \begin{bmatrix} 1 & -4 & -6 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -31 \\ -15 \\ 5 \\ 1 \end{bmatrix}$$

The matrix A whose columns are the given vectors has only 3 pivot positions. $\boxed{\text{Therefore these vectors do not form a basis}}$. In fact

$$-31\vec{v}_1 - 15\vec{v}_2 + 5\vec{v}_3 + \vec{v}_4 = \vec{0}.$$

2. **[3 points]** Given the matrix A and an echelon form of A, find a basis for $\text{Col}(A)$ and $\text{Nul}(A)$.

$$A = \begin{bmatrix} 75 & 383 & -281 & 32 & 329 \\ 10 & 51 & -37 & 4 & 44 \\ 38 & 194 & -142 & 16 & 167 \end{bmatrix} \sim \begin{array}{c} \begin{bmatrix} 1 & 5 & -3 & 0 & 6 \\ 0 & 1 & -7 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array}$$

Pivot columns are 1^{st}, 2^{nd}, and 5^{th} so a basis for $\text{Col}(A)$ is $\begin{bmatrix} 75 \\ 10 \\ 38 \end{bmatrix}, \begin{bmatrix} 383 \\ 51 \\ 194 \end{bmatrix}, \begin{bmatrix} 329 \\ 44 \\ 167 \end{bmatrix}$.

For $\underline{\text{Nul}(A)}$:

$$\begin{bmatrix} 1 & 5 & -3 & 0 & 6 \\ 0 & 1 & -7 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R2 \mathrel{+}= R3]{R1 \mathrel{+}= -6R3} \begin{bmatrix} 1 & 5 & -3 & 0 & 0 \\ 0 & 1 & -7 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R1 \mathrel{+}= -5R2} \begin{bmatrix} 1 & 0 & 32 & -20 & 0 \\ 0 & 1 & -7 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So $A\vec{x} = \vec{0}$ iff

$$\vec{x} = \begin{bmatrix} -32x_3 + 20x_4 \\ 7x_3 - 4x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -32 \\ 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 20 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_3, x_4 \in \mathbb{R}$$

So $\begin{bmatrix} -32 \\ 7 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 20 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is a basis for $\text{Nul}(A)$.

3. **[2 points]** Let H be the set of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4$ such that $x_1 + x_2 + x_3 + x_4 = 0$ and $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$. Find a basis for H.

$H = \text{Nul}(A)$, where $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \xrightarrow{R2 \stackrel{+}{=} -R1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{R1 \stackrel{+}{=} -R2} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Gen soln to $A\vec{x} = \vec{0}$: $\vec{x} = \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$. So a basis for H is $\boxed{\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$.

4. **[2 parts, 1 point each]** Subspaces.

(a) Let H_1 and H_2 be subspaces of $\mathbb{R}^n$. Prove that $H_1 \cap H_2$ is also a subspace of $\mathbb{R}^n$.

(i) Since $\vec{0} \in H_1$ and $\vec{0} \in H_2$, we have $\vec{0} \in H_1 \cap H_2$.

(ii). Suppose $\vec{x}, \vec{y} \in H_1 \cap H_2$. This means $\vec{x}, \vec{y} \in H_1$ and $\vec{x}, \vec{y} \in H_2$. Since H_1 is a subspace, $\vec{x} + \vec{y} \in H_1$. Similarly, $\vec{x} + \vec{y} \in H_2$. Since $\vec{x} + \vec{y}$ is in both H_1 and H_2, we have $\vec{x} + \vec{y} \in H_1 \cap H_2$.

(iii) Suppose $\vec{x} \in H_1 \cap H_2$ and c is a scalar. Since $\vec{x} \in H_1$ and H_1 is a subspace, we have $c\vec{x} \in H_1$. Similarly, $c\vec{x} \in H_2$. Therefore $c\vec{x} \in H_1 \cap H_2$.

Since $H_1 \cap H_2$ contains $\vec{0}$ and is closed under addition and scalar multiplication, $H_1 \cap H_2$ is a subspace.

(b) Give an example of subspaces H_1 and H_2 of $\mathbb{R}^2$ such that $H_1 \cup H_2$ is not a subspace.

Let $H_1 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ and $H_2 = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Now $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are both in $H_1 \cup H_2$ but the sum of these vectors, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, is not in $H_1 \cup H_2$. So $H_1 \cup H_2$ is not closed under vector addition and not a subspace.