Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[2 parts, 2 points each]** Decide whether the given transformation is linear. If the transformation is is linear, give the standard matrix. If the transformation is not linear, then explain why.

   (a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ |x_1 + x_2| \end{bmatrix}$ ↙ T

This is $\boxed{\text{not a linear transform.}}$ If $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, then

$T(\vec{u}) + T(\vec{v}) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ but

$T(\vec{u} + \vec{v}) = T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since $T(\vec{u}) + T(\vec{v}) \neq T(\vec{u} + \vec{v})$, we have that T is not a linear transform.

   (b) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} e^3 x_1 + \tan(\pi/5) x_2 \\ -x_1 \end{bmatrix}$. ↙ T

This is a $\boxed{\text{linear transform.}}$ Indeed, $T(\vec{x}) = \begin{bmatrix} e^3 & \tan(\frac{\pi}{5}) \\ -1 & 0 \end{bmatrix} \vec{x}$, so

the standard matrix for T is $\begin{bmatrix} e^3 & \tan(\frac{\pi}{5}) \\ -1 & 0 \end{bmatrix}$.

2. **[1 point]** Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transform, and let $\mathbf{v}_1, \ldots \mathbf{v}_p$ be vectors in $\mathbb{R}^n$. Show that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a linearly dependent set, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is linearly dependent.

<u>Pf.</u> Since $\vec{v}_1, \ldots, \vec{v}_p$ is linearly dependent, we have $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$ for some scalars $c_1, \ldots, c_p$ that are not all zero. Since T is linear, we have

$c_1 T(\vec{v}_1) + \cdots + c_p T(\vec{v}_p) = T(c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) = T(\vec{0}) = \vec{0}$. It follows that

$c_1 T(\vec{v}_1) + \cdots + c_p T(\vec{v}_p) = \vec{0}$ is a dependence equation for $T(\vec{v}_1), \ldots, T(\vec{v}_p)$ (recall that not all of $c_1, \ldots, c_p$ are zero), and so $T(\vec{v}_1), \ldots, T(\vec{v}_p)$ are linearly dependent.

3. **[2 parts, 2 points each]** Suppose that $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transform, let $\mathbf{u} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ and let $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. We know that T maps $\mathbf{u}$ to $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ and T maps $\mathbf{v}$ to $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$.

   (a) Find the image of $2\mathbf{u} - \mathbf{v}$ under T.

$$T(2\vec{u} - \vec{v}) = 2T(\vec{u}) - T(\vec{v}) = 2\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}}$$

   (b) If possible, then find $T(\mathbf{w})$, where $\mathbf{w} = \begin{bmatrix} 1 \\ -19 \end{bmatrix}$. If not possible, then explain why not.

Solve $\begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{w}$ for the scalars c_1, c_2:

$$\begin{bmatrix} -4 & 3 & 1 \\ 1 & -2 & -19 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -19 \\ -4 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -19 \\ 0 & -5 & -75 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -19 \\ 0 & 1 & 15 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & 15 \end{bmatrix}.$$

So $T(\vec{w}) = T(11\vec{u} + 15\vec{v}) = 11T(\vec{u}) + 15T(\vec{v}) = 11\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + 15\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

$$= \begin{bmatrix} 11 \\ -22 \\ 44 \end{bmatrix} + \begin{bmatrix} 30 \\ -15 \\ -15 \end{bmatrix} = \boxed{\begin{bmatrix} 41 \\ -37 \\ 29 \end{bmatrix}}$$

4. **[1 point]** Give a simple example of a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ such that the range of T is $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 + x_2 + x_3 = 0 \right\}$.

Many answers possible, for example $\boxed{T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ -x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$