Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[4 parts, 1 point each]** Determine whether the following vectors are linearly independent or linearly dependent. Justify your answer.

(a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ -6 \end{bmatrix}$

$\begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

Free variable

These are $\boxed{\text{linearly dependent}}$.

(b) $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -7 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 5 & -1 \\ 0 & 4 & 5 \\ 0 & 0 & -7 \\ 0 & 0 & 0 \end{bmatrix}$

No free variables, so these are $\boxed{\text{linearly independent}}$

(c) $\begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 5 & 5 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

free variable

These are $\boxed{\text{linearly dependent}}$

(d) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}$

Soln 1:

$\begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}$

Since this matrix has only 3 rows, it has at most 3 pivot positions. Some column is not a pivot column, so these are $\boxed{\text{linearly dependent}}$.

Soln 2: More vectors (4) than entries (3), so $\boxed{\text{lin. dependent}}$.

2. **[3 points]** Determine the values of h that make the following vectors linearly independent.

$$\begin{bmatrix} 2 \\ 14 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ h \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 35 \\ h+1 \end{bmatrix}$$

$\begin{bmatrix} 2 & -1 & 5 \\ 14 & h & 35 \\ 2 & -1 & h+1 \end{bmatrix}$ $\begin{matrix} R2 \pm (-7)R1 \\ R3 \pm (-1)R1 \end{matrix}$ $\begin{bmatrix} 2 & -1 & 5 \\ 0 & h+7 & 0 \\ 0 & 0 & h-4 \end{bmatrix}$ $\Rightarrow$ If $h \neq -7$ and $h \neq 4$, then all columns are pivot columns and so the vectors are linearly independent.

If $h=-7$, then swapping rows 2 and 3 puts the matrix into echelon form with the first and 3rd columns as pivots. So in this case the second column is not a pivot and the vectors are linearly dependent.

If $h=4$, then the first 2 columns are pivots and the 3rd is not, so the vectors are linearly dependent.

Therefore the values that make these vectors lin. independent are $\boxed{\text{all } h \text{ except } h=-7 \text{ and } h=4}$.

3. **[2 parts, 1 point each]** True/False. Justify your answers.

   (a) If the columns of A are linearly independent, then $A\mathbf{x} = \mathbf{0}$ has a unique solution.

$\boxed{\text{True}}$. $\{\vec{a}_1, \ldots, \vec{a}_p\}$ is lin. indep. $\iff$ $x_1\vec{a}_1 + \cdots + x_p\vec{a}_p = \vec{0}$ has only the trivial soln $x_1 = x_2 = \cdots = x_p = 0$

$\iff$ $A\vec{x} = \vec{0}$ has a unique soln.

Soln 3: Every column except $\vec{b}$ is a pivot in $[A\ \vec{b}]$, and so every col in A is a pivot. So $A\vec{x} = \vec{0}$ has no free vars.

   (b) If $A\mathbf{x} = \mathbf{b}$ has a unique solution for at least one vector $\mathbf{b}$, then the columns of A are linearly independent.

Soln #1 $\boxed{\text{True}}$. Suppose $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = \vec{u}$. Also, suppose $A\vec{y} = \vec{0}$. We have $A(\vec{u} + \vec{y}) = A\vec{u} + A\vec{y} = \vec{b} + \vec{0} = \vec{b}$. Since the only soln to $A\vec{x} = \vec{b}$ is $\vec{x} = \vec{u}$, it follows that $\vec{u} = \vec{u} + \vec{y}$ and therefore $\vec{y} = \vec{0}$. Since $A\vec{y} = \vec{0}$ implies $\vec{y} = \vec{0}$, the columns of A are lin. independent.

Soln #2: $\boxed{\text{True}}$. If the soln set to $A\vec{x} = \vec{b}$ is a single point, then this is a translation of the soln set of $A\vec{x} = \vec{0}$ which must also be a single point. Hence $A\vec{x} = \vec{0}$ has a unique soln.

4. **[1 point]** Prove that if $\mathbf{x}$ and $\mathbf{y}$ are linearly independent but $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\mathbf{z} \in \text{Span}\{\mathbf{x}, \mathbf{y}\}$.

Pf. Since $\{x, y, z\}$ is lin. dependent, we have $c_1 x + c_2 y + c_3 z = \vec{0}$ for some scalars c_1, c_2, c_3 that are not all zero. We claim that $c_3 \neq 0$. Indeed, if c_3 were zero, then our dependence equation becomes $c_1 x + c_2 y = \vec{0}$ for some scalars c_1 and c_2, not both of which are zero. But this is not possible, since this would be a dependence relation for $\{\vec{x}, \vec{y}\}$ and we know $\{\vec{x}, \vec{y}\}$ is linearly independent.

Since $c_3 \neq 0$, we may solve $c_1 x + c_2 y + c_3 z = \vec{0}$ for z:

$$c_3 \vec{z} = -c_1 x - c_2 y$$

$$\vec{z} = -\frac{c_1}{c_3} x - \frac{c_2}{c_3} y$$

It follows that $\vec{z} \in \text{Span}\{\vec{x}, \vec{y}\}$. ∎