Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[3 points]** Find a basis for the eigenspace corresponding to the given eigenvalue.

$$\begin{bmatrix} 1 & 0 & -3 \\ -6 & -2 & 6 \\ 0 & 0 & -2 \end{bmatrix}, \lambda = -2$$

$\text{Nul}(A - \lambda I) = \text{Nul}(A + 2I)$:

$$\begin{array}{c} \\ \begin{bmatrix} 3 & 0 & -3 \\ -6 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \sim \begin{array}{c} \\ \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \sim \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \atop \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2, x_3 free

$x_2 = 1,\ x_3 = 0$: $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$x_2 = 0,\ x_3 = 1$: $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

So the eigenspace is $\text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

2. **[2 parts, 2 points each]** Find the characteristic equation and eigenvalues with multiplicities of the following matrices.

(a) $\begin{bmatrix} -6 & 5 \\ -10 & 9 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} -6-\lambda & 5 \\ -10 & 9-\lambda \end{vmatrix} = 0$$

$(-6-\lambda)(9-\lambda) - 5(-10) = 0$

$\lambda^2 - 3\lambda - 54 + 50 = 0$

$\boxed{\lambda^2 - 3\lambda - 4 = 0}$

$(\lambda - 4)(\lambda + 1) = 0$

$\boxed{\lambda = -1, \quad \lambda = 4}$

(b) $\begin{bmatrix} -8 & 10 & 0 \\ -5 & 7 & 0 \\ 10 & -10 & 2 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} -8-\lambda & 10 & 0 \\ -5 & 7-\lambda & 0 \\ 10 & -10 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -8-\lambda & 10 \\ -5 & 7-\lambda \end{vmatrix} = (2-\lambda)\left[(-8-\lambda)(7-\lambda) - (-5)(10)\right] = 0$$

$(2-\lambda)[\lambda^2 + \lambda - 56 + 50] = 0$

$(2-\lambda)[\lambda^2 + \lambda - 6] = 0$

$(2-\lambda)(\lambda+3)(\lambda-2) = 0$

$\boxed{-(\lambda-2)^2(\lambda+3) = 0}$ or $\boxed{-\lambda^3 + \lambda^2 + 8\lambda - 12 = 0.}$

So $\lambda = 2$ (multiplicity 2), and $\lambda = -3$ (mult 1)

3. **[1 point]** Let A be an $n \times n$ matrix. Prove that if A and I_n are similar, then $A = I_n$.

If A and I_n are similar, then $A = P I_n P^{-1}$ for some matrix P.

It follows that $A = P I_n P^{-1} = P P^{-1} = I_n$. $\blacksquare$

4. **[4 parts, 0.5 points each]** True/False. In the following, A and B are $n \times n$ matrices. Justify your answers.

(a) If A and B have the same set of eigenvectors, then A and B are similar.

$\boxed{\text{FALSE}}$ If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, then both A and B have $\mathbb{R}^2 - \{\vec{0}\}$ as their set of eigenvectors, but A and B are not similar (by #3).

(b) If A and B are similar, then A and B have the same set of eigenvectors.

$\boxed{\text{FALSE}}$. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Let $B = PAP^{-1} = \overset{PA}{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}} \overset{P^{-1}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$ $= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. So A and B are similar but $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A and not B.

Alt: $\begin{tikzcd} \circ \arrow[r, "A"] \arrow[d, "P"'] & \circ \arrow[d, "P"] \\ \circ \arrow[r, "B"] & \circ \end{tikzcd}$ FALSE, the eigenvectors of A become eigenvectors of B when transformed by P.

(c) If A and B have the same characteristic polynomial, then A and B are similar.

$\boxed{\text{FALSE}}$ Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Both A and B have $(1-\lambda)^2$ as their characteristic polynomial but are not similar (by #3).

(d) If A and B are similar, then A and B have the same characteristic polynomial.

$\boxed{\text{TRUE}}$ If $B = PAP^{-1}$, then we have

$$f_B(\lambda) = \det(B - \lambda I) = \det(PAP^{-1} - \lambda I) = \det(PAP^{-1} - P(\lambda I)P^{-1})$$
$$= \det(P(A - \lambda I)P^{-1}) = \det(P)\det(A - \lambda I)\det(P^{-1}) = \det(PP^{-1})\det(A - \lambda I)$$
$$= 1 \cdot \det(A - \lambda I) = f_A(\lambda).$$

Alt: $\boxed{\text{True}}$, as we have seen in class.