Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[2 points]** Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, where $\mathbf{b}_1 = \begin{bmatrix} 5 \\ -1 \\ 4 \\ 2 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \end{bmatrix}$. Given $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 14 \\ 10 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{B}}$ if possible.

$$\begin{bmatrix} 5 & -2 & 2 & 3 \\ -1 & 3 & 0 & 12 \\ 4 & 1 & 3 & 14 \\ 2 & 1 & 1 & 10 \end{bmatrix} \xrightarrow{\substack{R1 \leftrightarrow R2 \\ R1 \doteq -1}} \begin{bmatrix} 1 & -3 & 0 & -12 \\ 5 & -2 & 2 & 3 \\ 4 & 1 & 3 & 14 \\ 2 & 1 & 1 & 10 \end{bmatrix} \xrightarrow{\substack{R2 \pm -5R1 \\ R3 \pm -4R1 \\ R4 \pm -2R1}} \begin{bmatrix} 1 & -3 & 0 & -12 \\ 0 & 13 & 2 & 63 \\ 0 & 13 & 3 & 62 \\ 0 & 7 & 1 & 34 \end{bmatrix} \xrightarrow{\substack{R3 \pm -R2 \\ R4 \doteq 2R2}} \begin{bmatrix} 1 & -3 & 0 & -12 \\ 0 & 13 & 2 & 63 \\ 0 & 0 & 1 & -1 \\ 0 & 14 & 2 & 68 \end{bmatrix}$$

$$\xrightarrow{R4 \pm -R2} \begin{bmatrix} 1 & -3 & 0 & -12 \\ 0 & 13 & 2 & 63 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 5 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R4} \begin{bmatrix} 1 & -3 & 0 & -12 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 13 & 2 & 63 \end{bmatrix} \xrightarrow{R4 \pm (-13)R2} \begin{bmatrix} 1 & -3 & 0 & -12 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix} \xrightarrow{R4 \pm (-2)R3}$$

$$\begin{bmatrix} 1 & -3 & 0 & -12 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \pm 3R2} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{So } [x]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}.$$

2. **[2 points]** Let A be a 5×8 matrix.

(a) What are the possible values for the dimension of the null space of A?

Note: $\text{rank}(A) + \dim(\text{Nul}(A)) = 8$. Since A is 5×8, the rank of A is in $\{0, 1, \dots, 5\}$. This means the dimension of $\text{Nul}(A)$ is in

$$\boxed{\{3, 4, 5, 6, 7, 8\}}.$$

(b) Suppose that the transform T given by $T(\mathbf{x}) = A\mathbf{x}$ is onto/surjective. Now what are the possible values for the dimension of the null space of A?

Since $x \mapsto Ax$ is surjective, we have $\text{col}(A) = \mathbb{R}^5$ so $\text{rank}(A) = \dim(\text{col}(A)) = 5$.

From the rank theorem, we get $\text{rank}(A) + \dim(\text{Nul}(A)) = 8$

$$\dim(\text{Nul}(A)) = 8 - \text{rank}(A) = 8 - 5 = \boxed{3.}$$

3. Compute the determinant of the following matrices.

(a) **[1 point]** $\begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix}$

$$3\cdot 3 - (-1)\cdot(2) = 9+2 = \boxed{11}$$

(b) **[1 point]** $\begin{bmatrix} 2 & -1 & 5 \\ 0 & 1 & -2 \\ 1 & 7 & -2 \end{bmatrix}$

$$2\begin{vmatrix} 1 & -2 \\ 7 & -2 \end{vmatrix} - 0\begin{vmatrix} \cdots \end{vmatrix} + 1\cdot\begin{vmatrix} -1 & 5 \\ 1 & -2 \end{vmatrix} = 2\cdot(-2-(-2)(7)) + 1((-1)(-2) - (1)(5))$$

$$= 2(12) + 1(-3) = \boxed{21}$$

(c) **[2 points]** $\begin{bmatrix} \overset{+}{2} & \overset{-}{6} & \overset{+}{0} & -5 \\ 4 & 1 & \overset{-}{3} & 8 \\ 0 & 5 & 0 & 0 \\ 3 & -2 & 0 & 1 \end{bmatrix} = (-3)\begin{vmatrix} \overset{+}{2} & \overset{-}{6} & -5 \\ 0 & \overset{+}{5} & 0 \\ 3 & -2 & 1 \end{vmatrix} = (-3)(5)\begin{vmatrix} 2 & -5 \\ 3 & 1 \end{vmatrix}$

$$= (-15)((2)(1) - (-5)(3)) = (-15)(17) = -(100+50+70+35) = \boxed{-255}$$

(d) **[2 points]** $\begin{bmatrix} 1 & 3 & 1 & 5 \\ -2 & -4 & -3 & -6 \\ 1 & 3 & 2 & 8 \\ 3 & 9 & 3 & 12 \end{bmatrix}$ (Hint: use row reduction)

$$\underset{A}{\begin{bmatrix} 1 & 3 & 1 & 5 \\ -2 & -4 & -3 & -6 \\ 1 & 3 & 2 & 8 \\ 3 & 9 & 3 & 12 \end{bmatrix}} \xrightarrow{\substack{R2 \pm 2R1 \\ R3 \pm -1R1 \\ R4 \pm -3R1}} \underset{B}{\begin{bmatrix} 1 & 3 & 1 & 5 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix}}$$

$$\det(B) = (1)(2)(1)(-3) = -6$$

$$\det(A) = 1\cdot\det(B) = \boxed{-6}$$

only row replacement.