Solutions

Directions:

1. Section: Math251
2. Write your name with one character in each box below.
3. Show all work. No credit for answers without work.
4. This assessment is closed book and closed notes. You may not use electronic devices, including calculators, laptops, and cell phones.

Academic Integrity Statement: I will complete this work on my own without assistance, knowing or otherwise, from anyone or anything other than the instructor. I will not use any electronic equipment or notes (except as permitted by an existing official, WVU-authorized accommodation).

Signature: ______________________________

1. **[2 parts, 6 points each]** A particle travels in a straight line at constant speed. At time $t = 0$, the particle is at $P(2, -1, 4)$, and at time $t = 1$, the particle is at $Q(5, 0, 3)$.

   (a) Find parametric equations for the position of the particle at time t.

$$\vec{v} = \vec{PQ} = \langle 5-2,\ 0-(-1),\ 3-4 \rangle = \langle 3, 1, -1 \rangle.$$

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

$$\vec{r} = \langle 2, -1, 4 \rangle + t\langle 3, 1, -1 \rangle$$

Parametric Equations:

$$\boxed{\begin{aligned} x(t) &= 2 + 3t \\ y(t) &= -1 + t \\ z(t) &= 4 - t \end{aligned}}$$

   (b) Find the speed of the particle.

$$\text{Speed} = |\vec{r}\,'(t)| = |\langle 3, 1, -1 \rangle| = \left(3^2 + 1^2 + (-1)^2\right)^{\frac{1}{2}} = \boxed{\sqrt{11}}$$

2. **[6 points]** A particle starts at $P(1, 2, 3)$ at time $t = 0$ and has velocity vector $\mathbf{r}'(t) = \langle 2t, \sin t, t^2 \rangle$. Find the position vector $\mathbf{r}(t)$.

$$\vec{r}(t) = \int \vec{r}\,'(t)\,dt = \int \langle 2t, \sin t, \ln t \rangle\,dt = \langle t^2, -\cos t, \tfrac{1}{3}t^3 \rangle + C$$

$\vec{r}(0) = \langle 1, 2, 3 \rangle$:

$$\langle 1, 2, 3 \rangle = \langle 0^2, -\cos 0, \tfrac{1}{3}0^3 \rangle + C$$

$$C = \langle 1, 2, 3 \rangle - \langle 0, -1, 0 \rangle = \langle 1, 3, 3 \rangle$$

$$\vec{r}(t) = \langle t^2, -\cos t, \tfrac{1}{3}t^3 \rangle + \langle 1, 3, 3 \rangle$$

$$= \boxed{\langle t^2 + 1,\ 3 - \cos t,\ 3 + \tfrac{1}{3}t^3 \rangle}$$

3. **[8 points]** A particle has position vector $\mathbf{r}(t) = \langle t^2, t, t^2 - 3t \rangle$. Find the minimum speed of the particle.

$$\frac{dv}{dt} = \frac{d}{dt}\left[\,|\vec{r}\,'(t)|\,\right]$$

$$\frac{d}{dt}\left[(\vec{r}\,' \cdot \vec{r}\,')^{\frac{1}{2}}\right]$$

$$\frac{1}{2}(\vec{r}\,' \cdot \vec{r}\,')^{-\frac{1}{2}} \frac{d}{dt}\left[\vec{r}\,' \cdot \vec{r}\,'\right]$$

$$\frac{1}{2} \cdot \frac{1}{|\vec{r}\,'|} \cdot (\vec{r}\,'' \cdot \vec{r}\,' + \vec{r}\,' \cdot \vec{r}\,'')$$

$$\cancel{\frac{1}{2}} \cdot \frac{1}{|\vec{r}\,'|} \cdot (\cancel{2}\,\vec{r}\,' \cdot \vec{r}\,'')$$

$$\frac{1}{|\vec{r}\,'|}(\vec{r}\,' \cdot \vec{r}\,'')$$

$$\vec{r}\,' = \langle 2t, 1, 2t-3 \rangle$$

$$\vec{r}\,'' = \langle 2, 0, 2 \rangle$$

$$\frac{dv}{dt} = \frac{1}{|\vec{r}\,'|}(\vec{r}\,' \cdot \vec{r}\,'')$$

$$= \frac{1}{|\vec{r}\,'|}\left((2t)(2) + (1)(0) + (2t-3)(2)\right)$$

$$= \frac{1}{|\vec{r}\,'|}(8t - 6)$$

1

$\frac{dv}{dt} = 0$: $\quad \frac{1}{|\vec{r}\,'|}(8t-6) = 0$

$8t = 6$, $\quad t = 3/4$

Sign of $\frac{dv}{dt}$: $\quad -\quad 0 \quad +$ (at $3/4$)

So the minimum speed occurs at $t = \frac{3}{4}$.

$$\text{Min speed} = |\vec{r}\,'(\tfrac{3}{4})|$$

$$= |\langle \tfrac{3}{2}, 1, \tfrac{3}{2} - 3 \rangle|$$

$$= |\tfrac{1}{2}\langle 3, 2, -3 \rangle|$$

$$= \tfrac{1}{2}\left(3^2 + 2^2 + (-3)^2\right)^{1/2} = \boxed{\tfrac{1}{2}\sqrt{22}}$$

4. **[3 parts, 8 points each]** Let $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ and note that at $t = 1$, the normal vector $\mathbf{N}$ equals $\frac{1}{\sqrt{2\cdot 7\cdot 19}} \langle -11, -8, 9 \rangle$.

(a) Find the tangent vector $\mathbf{T}$ and binormal vector $\mathbf{B}$ at $t = 1$.

$T = \dfrac{\vec{r}\,'}{|\vec{r}\,'|}$

$\vec{r}\,' = \langle 1, 2t, 3t^2 \rangle$

$|\vec{r}\,'| = (1 + 4t^2 + 9t^4)^{1/2}$

$T(1) = \dfrac{\vec{r}\,'(1)}{|\vec{r}\,'(1)|} = \boxed{\dfrac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle}$

$B = T \times N = \dfrac{1}{\sqrt{2\cdot 7}} \langle 1, 2, 3 \rangle \times \dfrac{1}{\sqrt{2\cdot 7\cdot 19}} \langle -11, -8, 9 \rangle$

$= \dfrac{1}{\sqrt{2\cdot 7}} \cdot \dfrac{1}{\sqrt{2\cdot 7\cdot 19}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -11 & -8 & 9 \end{vmatrix} = \dfrac{1}{14\sqrt{19}} \left[\hat{i}(18 + 24) - \hat{j}(9 + 33) + \hat{k}(-8 + 22) \right]$

$= \dfrac{1}{14\sqrt{19}} \langle 42, -42, 14 \rangle = \dfrac{\cancel{14}}{\cancel{14}\sqrt{19}} \langle 3, -3, 1 \rangle$

$= \boxed{\dfrac{1}{\sqrt{19}} \langle 3, -3, 1 \rangle}$.

(b) Find vector equations for the normal plane and osculating plane at time $t = 1$.

Normal Plane: T is the normal vector.

$T \cdot (\vec{r} - \vec{r}_0) = 0$

vector form

$\boxed{\dfrac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle \cdot (\overbrace{\langle x, y, z \rangle}^{\vec{r}} - \langle 1, 1, 1 \rangle) = 0}$

$\langle 1, 2, 3 \rangle \cdot \langle x - 1, y - 1, z - 1 \rangle = 0$

$\boxed{(x - 1) + 2(y - 1) + 3(z - 1) = 0}$ standard form

Osculating Plane: B is the normal vector

$B \cdot (\vec{r} - \vec{r}_0) = 0$

vector form

$\boxed{\dfrac{1}{\sqrt{19}} \langle 3, -3, 1 \rangle \cdot (\overbrace{\langle x, y, z \rangle}^{\vec{r}} - \langle 1, 1, 1 \rangle) = 0}$

$\boxed{3(x - 1) - 3(y - 1) + (z - 1) = 0}$

standard form

(c) Find the radius and center of the osculating circle at $t = 1$.

Radius $= \dfrac{1}{K}$

$K = \dfrac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$

$\vec{r}\,' = \langle 1, 2t, 3t^2 \rangle$

$\vec{r}\,'' = \langle 0, 2, 6t \rangle$

$|\vec{r}\,'| = (1 + 4t^2 + 9t^4)^{1/2}$

$\vec{r}\,'(1) = \langle 1, 2, 3 \rangle$

$\vec{r}\,''(1) = \langle 0, 2, 6 \rangle$

$|\vec{r}\,'(1)| = \sqrt{14}$

$\vec{r}\,'(1) \times \vec{r}\,''(1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix}$

$= \hat{i}(12 - 6) - \hat{j}(6 - 0) + \hat{k}(2 - 0)$

$= \langle 6, -6, 2 \rangle$.

So $K = \dfrac{|\langle 6, -6, 2 \rangle|}{(14)^{3/2}} = \dfrac{(36 + 36 + 4)^{1/2}}{(14)^{3/2}}$

$= \dfrac{(76)^{1/2}}{(14)^{3/2}} = \left[\dfrac{4 \cdot 19}{2^3 \cdot 7^3} \right]^{1/2} = \left[\dfrac{19}{2 \cdot 7^3} \right]^{1/2}$

Radius $= \dfrac{1}{K} = \left(\dfrac{2 \cdot 7^3}{19} \right)^{1/2} = \boxed{\dfrac{7\sqrt{14}}{\sqrt{19}}}$

Center $= \vec{r}(1) + (\text{Radius}) \vec{N}(1)$

$= \langle 1, 1, 1 \rangle + \dfrac{7\cancel{\sqrt{14}}}{\sqrt{19}} \cdot \dfrac{1}{\sqrt{\cancel{2\cdot 7}\cdot 19}} \cdot \langle -11, -8, 9 \rangle$

$= \langle 1, 1, 1 \rangle + \dfrac{7}{19} \langle -11, -8, 9 \rangle$

$= \dfrac{1}{19} \left(\langle 19, 19, 19 \rangle + \langle -77, -56, 63 \rangle \right)$

$= \boxed{\dfrac{1}{19} \langle -58, -37, 82 \rangle}$

5. Let $f(x, y) = x\sin(2x + 3y)$.

(a) **[12 points]** Find all first and second partials of $f(x, y)$.

$f_x = \frac{\partial}{\partial x}\left[x\sin(2x+3y)\right]$

$= \sin(2x+3y) + 2x\cos(2x+3y)$

$f_y = 3x\cos(2x+3y)$

$f_{xx} = 2\cos(2x+3y) + 2\cos(2x+3y) - 4x\sin(2x+3y)$

$f_{xy} = 3\cos(2x+3y) - 2x\sin(2x+3y)\cdot 3 = 3\cos(2x+3y) - 6x\sin(2x+3y)$

$f_{yx} = 3\cos(2x+3y) - 3x\sin(2x+3y)\cdot 2 = 3\cos(2x+3y) - 6x\sin(2x+3y)$

$f_{yy} = -3x\sin(2x+3y)\cdot 3 = -9x\sin(2x+3y)$

(b) **[6 points]** Find a vector equation for the tangent plane to the graph of $f(x, y)$ at $(\underbrace{\pi/4}_{x_0}, \underbrace{0}_{y_0}, \underbrace{\pi/4}_{z_0})$.

$z - z_0 = f_x \cdot \Delta x + f_y \cdot \Delta y$

$z - \frac{\pi}{4} = 1\left(x - \frac{\pi}{4}\right) + 0\cdot(y - 0)$

$z - \frac{\pi}{4} = x - \frac{\pi}{4}$

$\boxed{z = x}$

$f_x\left(\frac{\pi}{4}, 0\right) = \sin\left(2\cdot\frac{\pi}{4} + 0\right) + 2\frac{\pi}{4}\cos\left(\cancelto{0}{\frac{\pi}{2}}\right) = 1$

$f_y\left(\frac{\pi}{4}, 0\right) = 3\cdot\frac{\pi}{4}\cos\left(2\frac{\pi}{4} + 0\right) = 0$

6. **[2 parts, 6 points each]** The temperature at a point (x, y, z) is given by the function $T(x, y, z) = x^2y + 3z^2$.

(a) Find the rate of change in temperature if we start at $(1, -3, 2)$ and move toward the origin.

$\vec{v} = \langle 0-1, 0-(-3), 0-2\rangle$

$= \langle -1, 3, -2\rangle$

$D_u[T] = \nabla T \cdot \frac{\vec{v}}{|\vec{v}|}$

$= \langle 2xy, x^2, 6z\rangle \cdot \frac{1}{\sqrt{14}}\langle -1, 3, -2\rangle$

$= \frac{1}{\sqrt{14}}\langle 2(1)(-3), 1^2, 6\cdot 2\rangle \cdot \langle -1, 3, -2\rangle$

$= \frac{1}{\sqrt{14}}\langle -6, 1, 12\rangle \cdot \langle -1, 3, -2\rangle$

$= \frac{1}{\sqrt{14}}(6 + 3 - 24) = \boxed{\frac{15}{\sqrt{14}}}$

(b) Starting at $(1, -3, 2)$, we want to increase the temperature is quickly as possible. In which direction should we travel, and what is the rate of increase?

Move in direction of $\nabla T(1,-3,2) = \langle 2xy, x^2, 6z\rangle\Big|_{(1,-3,2)} = \boxed{\langle -6, 1, 12\rangle}$

Rate of increase: $D_u[T] = \nabla T \cdot \frac{\nabla T}{|\nabla T|} = \frac{|\nabla T|^2}{|\nabla T|} = |\nabla T| = |\langle -6, 1, 12\rangle| = (36 + 1 + 144)^{1/2}$

$= \boxed{\sqrt{181}}$

7. **[10 points]** Let $f(x,y,z) = \ln(x^2+y^2+z^2)$ and let $x(s,t) = 2s+3t$, $y(s,t) = st$, and $z(s,t) = s^2e^t$. Compute $\frac{\partial f}{\partial s}$ when $s=2$ and $t=0$.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$= \frac{2x}{x^2+y^2+z^2}\cdot 2 + \frac{2y}{x^2+y^2+z^2}\cdot t + \frac{2z}{x^2+y^2+z^2}\cdot 2e^t s$$

$$= \frac{4x+2yt+4zse^t}{x^2+y^2+z^2} = \frac{4\cdot4+2\cdot0\cdot0+4\cdot4\cdot2\cdot e^0}{4^2+0^2+4^2} = \frac{4^2(1+0+2)}{2\cdot4^2} = \frac{3\cdot\cancel{4^2}}{2\cdot\cancel{4^2}} = \boxed{\frac{3}{2}}.$$

$x(2,0) = 2\cdot2+3\cdot0 = 4$
$y(2,0) = 2\cdot0 = 0$
$z(2,0) = 2^2\cdot e^0 = 4$

8. **[10 points]** Find and classify the critical points of $f(x,y) = xy^2 - x^2y + 4x - 4y$ as local minima, local maxima, or saddle points.

$f_x = y^2 - 2xy + 4$

$f_y = 2xy - x^2 - 4$

Find Critical Pts: $f_x = 0$ and $f_y = 0$

$y^2 - 2xy + 4 = 0$
$-x^2 + 2xy - 4 = 0$

$y^2 - x^2 = 0$

$(y+x)(y-x) = 0$

$y+x=0$ or $y-x=0$

$y=-x$ or $y=x$

$(-x)^2 - 2x(-x) + 4 = 0$
$x^2 + 2x^2 = -4$
$3x^2 = -4$
No Soln

$x^2 - 2x^2 + 4 = 0$
$x^2 = 4$
$x=-2,\ x=2$
$y=-2,\ y=2$

• Critical points: $(-2,-2)$ and $(2,2)$.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{vmatrix} = \begin{vmatrix} -2y & 2y-2x \\ 2y-2x & 2x\end{vmatrix} = -4xy - (2y-2x)^2$$

$$= -4xy - 4(y-x)^2 = -4[xy + (y-x)^2]$$

$$D(-2,-2) = -4[(-2)(-2) + (-2-(-2))^2] = -4\cdot4 < 0$$

$\Rightarrow$ $\boxed{(-2,-2) \text{ is a saddle point.}}$

$$D(2,2) = -4[(2)(2) + 0^2] = -4\cdot4 < 0$$

$\Rightarrow$ $\boxed{(2,2) \text{ is also a saddle point.}}$