Directions:

1. Section: Math251 007
2. Write your name with one character in each box below.
3. Show all work. No credit for answers without work.

1. **[10 points]** Use matrices and row operations to give a simple description for the set of solutions to the following system.

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & - & 9x_3 & = & 5 \\ 3x_1 & + & x_2 & - & 13x_3 & = & 8 \end{array}$$

$$\begin{bmatrix} 1 & -2 & -9 & 5 \\ 3 & 1 & -13 & 8 \end{bmatrix} \xrightarrow{R2 \pm (-3)R1} \begin{bmatrix} 1 & -2 & -9 & 5 \\ 0 & 7 & 14 & -7 \end{bmatrix} \xrightarrow{R2 \div \frac{1}{7}} \begin{bmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -1 \end{bmatrix} \xrightarrow{R1 \pm 2R2} \begin{bmatrix} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

$x_1 = 5x_3 + 3$

$x_2 = -2x_3 - 1$

$x_3 = x_3$

So the soln set is $\left\{ \begin{bmatrix} 3+5s \\ -1-2s \\ s \end{bmatrix} : s \in \mathbb{R} \right\}$

2. Let S be the sphere $x^2 + 3x + y^2 - 2y + z^2 + 4z = 13$.

(a) **[10 points]** Find the center and radius of S.

$(x+\frac{3}{2})^2 - \frac{9}{4} + (y-1)^2 - 1 + (z+2)^2 - 4 = 13$

$(x-(-\frac{3}{2}))^2 + (y-1)^2 + (z-(-2))^2 = 18 + \frac{9}{4} = 20 + \frac{1}{4} = \frac{81}{4} = \left(\frac{9}{2}\right)^2 \quad (*)$

So the center is $\boxed{(-\frac{3}{2}, 1, -2)}$ and the radius is $\boxed{\frac{9}{2}}$.

(b) **[6 points]** Give a full geometric description of the curve obtained at the intersection of S and the plane $z = 0$.

Set $z=0$ in $(*)$:

$(x-(-\frac{3}{2}))^2 + (y-1)^2 + (2)^2 = \frac{81}{4}$

$(x-(-\frac{3}{2}))^2 + (y-1)^2 = \frac{81}{4} - 4 = \frac{81}{4} - \frac{16}{4} = \frac{65}{4} = \left(\frac{\sqrt{65}}{2}\right)^2$

$(x-(-\frac{3}{2}))^2 + (y-1)^2 = \left(\frac{\sqrt{65}}{2}\right)^2$

This is the circle in the $z=0$ plane centered at $(\frac{3}{2}, 1, 0)$ with radius $\frac{\sqrt{65}}{2}$.

3. **[4 parts, 4 points each]** Let $\mathbf{a} = \langle 1, -5, 2\rangle$ and $\mathbf{b} = \langle 1, -1, -2\rangle$. Compute the following.

(a) $3\mathbf{a} - 2\mathbf{b}$

$\langle 3, -15, 6\rangle - \langle 2, -2, -4\rangle$

$= \boxed{\langle 1, -13, 10\rangle}$

(b) $|\mathbf{a}|$

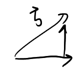
$= \sqrt{1^2 + (-5)^2 + 2^2} = \boxed{\sqrt{30}}$

(c) $\mathbf{a} \cdot \mathbf{b}$

$\vec{a} \cdot \vec{b} = (1)(1) + (-5)(-1) + (2)(-2)$

$= 1 + 5 - 4 = \boxed{2}$

(d) $\text{proj}_{\mathbf{a}}\mathbf{b}$



$\dfrac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a} = \dfrac{2}{30} \langle 1, -5, 2\rangle = \boxed{\left\langle \frac{1}{15}, \frac{-1}{3}, \frac{2}{15} \right\rangle}$

4. **[6 points]** Find numbers h and k such that $\langle 3, 2, -5\rangle$ and $\langle h, -1, k\rangle$ are parallel.

Need α such that $\alpha\langle 3, 2, -5\rangle = \langle h, -1, k\rangle$.

2^{nd} component requires $2\alpha = -1$, so $\alpha = -\frac{1}{2}$.

$h = 3\alpha = \dfrac{-3}{2}$

$k = -5\alpha = -5(-\tfrac{1}{2}) = \dfrac{5}{2}$

So $(h, k) = \boxed{\left(\frac{-3}{2}, \frac{5}{2}\right)}$.

5. Let $\mathbf{a} = \langle 2, 1, 0\rangle$ and $\mathbf{b} = \langle -1, 3, 5\rangle$.

(a) **[8 points]** Find the angle between $\mathbf{a}$ and $\mathbf{b}$. Leave your answer in terms of inverse trigonometric functions.

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$, $\quad \cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$ $\quad \dfrac{(2)(-1) + (1)(3) + (0)(5)}{\sqrt{2^2 + 1^2 + 0^2} \cdot \sqrt{(-1)^2 + 3^2 + 5^2}} = \dfrac{1}{\sqrt{5} \cdot \sqrt{35}} = \dfrac{1}{\sqrt{5^2 \cdot 7}} = \dfrac{1}{5\sqrt{7}}$

So $\theta = \cos^{-1}\left(\dfrac{1}{5\sqrt{7}}\right)$

(b) **[2 points]** Is the angle between $\mathbf{a}$ and $\mathbf{b}$ acute, obtuse, or right? Explain.

Since $\vec{a} \cdot \vec{b} > 0$, the angle between $\vec{a}$ and $\vec{b}$ is $\boxed{\text{acute}}$.

6. Let T be the triangle with vertices $\overset{P}{(2,1,0)}$, $\overset{Q}{(3,1,4)}$ and $\overset{R}{(1,5,1)}$.

   (a) **[10 points]** Find a simple equation for the plane containing T.

$\vec{a} = \overrightarrow{PQ} = \langle 3-2, 1-1, 4-0\rangle = \langle 1, 0, 4\rangle$

$\vec{b} = \overrightarrow{PR} = \langle 1-2, 5-1, 1-0\rangle = \langle -1, 4, 1\rangle$

$$\vec{n} = \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 4 \\ -1 & 4 & 1 \end{vmatrix} = \hat{i}\begin{vmatrix} 0 & 4 \\ 4 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 4 \\ -1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 0 \\ -1 & 4 \end{vmatrix}$$

$= -16\hat{i} - \hat{j}5 + 4\hat{k}$

$= \langle -16, -5, 4\rangle$

$\vec{n}\cdot(\vec{r}-\vec{r}_0) = 0$

$\langle -16, -5, 4\rangle \cdot \langle x-2, y-1, z\rangle = 0$

$-16(x-2) - 5(y-1) + 4z = 0$

$-16x + 32 - 5y + 5 + 4z = 0$

$-16x - 5y + 4z = -37$

$\boxed{16x + 5y - 4z = 37}$

   (b) **[6 points]** Find the area of T.

$\text{area}(T) = \frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}|\langle -16, -5, 4\rangle|$

$= \frac{1}{2}\sqrt{(-16)^2 + (-5)^2 + 4^2}$

$= \sqrt{\frac{1}{4}\left[(16)^2 + (5)^2 + 4^2\right]}$

$= \sqrt{8^2 + (\frac{5}{2})^2 + 2^2}$

$= \sqrt{64 + \frac{25}{4} + 4} = \sqrt{\frac{256 + 25 + 16}{4}}$

$= \sqrt{\frac{256 + 41}{4}} = \sqrt{\frac{297}{4}} = \boxed{\frac{\sqrt{297}}{2}}$

7. **[10 points]** If the following lines intersect, then find the point of intersection. If the lines do not intersect, determine if they are parallel or skew.

$$\mathbf{r}_1 = \langle 9, 3, -16\rangle + t\,\langle 1, 1, 4\rangle \qquad \mathbf{r}_2 = \langle 5, 14, -5\rangle + t\,\langle -2, 3, 1\rangle$$

$\langle 9 + t_1, 3 + t_1, -16 + 4t_1\rangle = \langle 5 - 2t_2, 14 + 3t_2, -5 + t_2\rangle$

$t_1 + 9 = 5 - 2t_2$

$t_1 + 3 = 14 + 3t_2$

$4t_1 - 16 = t_2 - 5$

$t_1 + 2t_2 = -4$

$t_1 - 3t_2 = 11$

$4t_1 - t_2 = 11$

$$\begin{bmatrix} 1 & 2 & -4 \\ 1 & -3 & 11 \\ 4 & -1 & 11 \end{bmatrix} \xrightarrow[R3 \doteq (-4)R1]{R2 \doteq (-1)R1} \begin{bmatrix} 1 & 2 & -4 \\ 0 & -5 & 15 \\ 0 & -9 & 27 \end{bmatrix}$$

$$\begin{matrix} R2 \doteq -\frac{1}{5} \\ \\ R3 \doteq -\frac{1}{9} \end{matrix} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow[R1 \doteq (-2)R2]{R3 \doteq (-1)R2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

The lines intersect with $t_1 = 2$ and $t_2 = -3$ at point $\langle 9, 3, -16\rangle + 2\langle 1, 1, 4\rangle = \boxed{\langle 11, 5, -8\rangle}$.

8. Consider the surface $\frac{x^2}{4} + y + \frac{z^2}{9} = 1$.

   (a) **[6 points]** Describe the axis-aligned traces.

$\frac{x^2}{4^2} + \frac{z^2}{6^2} = 1$

$x = k$:

$\frac{k^2}{4} + y + \frac{z^2}{9} = 1$

$y = -\frac{z^2}{9} + 1 - \frac{k^2}{4}$

parabola opening in $-y$ direction

$y = k$

$\frac{x^2}{4} + \frac{z^2}{9} = 1 - k$

Ellipse when $1 - k > 0$

$k < 1$

No soln when $k > 1$

$z = k$:

$\frac{x^2}{4} + y + \frac{k^2}{9} = 1$

$y = -\frac{x^2}{4} + 1 - \frac{k^2}{9}$

Parabola opening in $-y$ direction

   (b) **[10 points]** Classify the surface and sketch it, labeling any significant points.

This is an elliptic paraboloid with vertex at $(0,1,0)$, opening in the negative y direction.