Name: ______________________________

Directions: Show all work. No credit for answers without work.

1. **[15 points]** Use cofactor expansion to compute the determinant of the following matrix as efficiently as possible.

$$\begin{bmatrix} 0 & 0 & 3 & 0 & 1 \\ 0 & 2 & 5 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 7 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. **[2 parts, 5 points each]** Let A be an invertible $n \times n$-matrix.

   (a) What can you say about the rank of A?

   (b) Suppose that C is an $n \times n$ matrix obtained from A by changing one row of A. What can you say about the rank of C? Why?

3. **[25 points]** Using standard techniques from class, diagonalize the following matrix A. That is, if possible, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. The matrix P^{-1} need not be computed explicitly. If diagonalization is not possible, then explain why.

$$\begin{bmatrix} 2 & 3 & -3 \\ -3 & -4 & 3 \\ -3 & -3 & 2 \end{bmatrix}$$

4. A casino operates a slot machine, where each play costs 1 dollar. The machine either returns no cash (a loss), returns 1 dollar (a tie), or returns 5 dollars (a win). The chances of an outcome depend on the previous outcome. If the previous outcome was a loss, then the next outcome has a 50% chance of being another loss, a 30% chance of being a tie, and a 20% chance of being a win. If the previous outcome was a tie, then the next outcome has a 90% chance of being a loss and a 10% chance of being another tie. If the previous outcome is a win, then the next outcome has a 100% chance of being a loss.

   (a) [**5 points**] Draw a state diagram that models the slot machine.

   (b) [**5 points**] Give the stochastic matrix P for the corresponding Markov chain.

   (c) [**10 points**] Find the steady state vector for P.

   (d) [**5 points**] On average, how much money does the player lose on each play?

5. **[6 parts, 2 points each]** True/False. Justify your answer.

   (a) If A has a strictly dominant eigenvalue λ, then for most vectors $\mathbf{x}_0$, the sequence $\mathbf{x}_0, \mathbf{x}_1, \ldots$ defined by $\mathbf{x}_k = A^k\mathbf{x}_0$ approaches an eigenvector for λ.

   (b) If A and B are $n \times n$ matrices, then $\det(A + B) = \det(A) + \det(B)$.

   (c) If A and B are similar matrices, then A and B have the same eigenvalues with the same multiplicities.

   (d) Let A be a matrix with 7 rows and 10 columns. If the null space of A has dimension 4, then the column space of A has dimension 3.

   (e) If A is a square matrix and r is a scalar, then A is similar to rA.

   (f) If A is diagonalizable, then so is $A^2 + A$.

6. **[5 points]** Find the distance between the complex-valued vectors $\begin{bmatrix} 3+i \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1+2i \\ i \end{bmatrix}$.

7. **[8 points]** Find a unit vector in the direction of $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$.