Name: Solutions

Directions: Show all work. No credit for answers without work.

1. A fair die is rolled 3 times. Let A be the event that the first roll is strictly smaller than the second (so $(4,5,1) \in A$ but $(4,4,1) \notin A$). Let B be the event that all three rolls are distinct.

   (a) **[1 point]** What is the probability space Ω? What is $|\Omega|$?

   $$\Omega = \{1, \dots, 6\}^3$$

   $$|\Omega| = 6^3 = 216$$

   (b) **[2 points]** Determine $\Pr(A)$ and $\Pr(B)$.

   Choose pair of distinct numbers for first two rolls

   Choose any number in $\{1,\dots,6\}$ for 3^{rd} roll

   $$\Pr(A) = \frac{|A|}{|\Omega|} = \frac{\binom{6}{2}\cdot 6}{6^3} = \frac{\frac{\cancel{6}\cdot 5}{2}\cdot}{6^{\cancel{2}}} = \frac{5}{2\cdot 6} = \boxed{\frac{5}{12}}$$

   $$\Pr(B) = \frac{|B|}{|\Omega|} = \frac{6\cdot5\cdot4}{6^3} = \frac{5\cdot4}{6\cdot6} = \frac{5}{3\cdot3} = \boxed{\frac{5}{9}}$$

   (c) **[2 points]** Determine $\Pr(A|B)$ and $\Pr(B|A)$.

   Count $A\cap B$ using rule of product: ① Choose numbers for first two rolls ($\binom{6}{2}$ options)
   ② Choose 3^{rd} roll different from first two (4 options)

   $$|A\cap B| = \binom{6}{2}\cdot 4 = \frac{6\cdot5}{2}\cdot 4 = 3\cdot5\cdot4$$

   $$\Pr(A|B) = \frac{\Pr(A\cap B)}{\Pr(B)} = \frac{(3\cdot\cancel{5}\cdot4)/6^3}{\cancel{5}/9} = \frac{\cancel{3}\cdot4\cdot\cancel{9}}{6\cdot6\cdot6} = \frac{\cancel{4}}{2\cdot2\cdot\cancel{2}} = \boxed{\frac{1}{2}}$$

   $$\Pr(B|A) = \frac{\Pr(A\cap B)}{\Pr(A)} = \frac{(3\cdot\cancel{5}\cdot4)/6^3}{\cancel{5}/12} = \frac{\cancel{12}\cdot\cancel{3}\cdot4}{\cancel{6}\cdot\cancel{6}\cdot6} = \boxed{\frac{2}{3}}$$

   (d) **[1 point]** Are the events A and B pos. correlated, neg. correlated, or independent?

   Since $\Pr(A|B) = \frac{1}{2} > \frac{5}{12} = \Pr(A)$, these events are

   $\boxed{\text{positively correlated}}$

2. **[2 parts, 2 points each]** Recall that a standard deck of cards has one card for each rank/suit pair, where the ranks are [ace, 2 through 10, jack, queen, king], and the suits are [clubs, hearts, diamonds, spades]. A 5-card poker hand is dealt from a freshly shuffled deck.

   (a) What is the probability that hand has no spades?

Ω: all poker hands.

A: hand has no spades

$$\Pr(A) = \frac{|A|}{|\Omega|} = \frac{\binom{52-13}{5}}{\binom{52}{5}} = \boxed{\frac{\binom{39}{5}}{\binom{52}{5}}} \approx 0.2215$$

   (b) It is revealed that the hand has all distinct ranks. Now what is the probability that the hand has no spades?

B: hand has all distinct ranks.

$$|B| = \underbrace{\binom{13}{5}}_{\text{choose ranks}} \cdot \underbrace{4^5}_{\text{for each chosen rank, choose a suit}}$$

$A \cap B$: cards with no spades and distinct ranks; $|A \cap B| = \underbrace{\binom{13}{5}}_{\text{choose ranks}} \cdot \underbrace{3^5}_{\text{choose suits (no spades)}}$

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\dfrac{\cancel{\binom{13}{5}} \cdot 3^5}{\cancel{\binom{52}{5}}}}{\dfrac{\cancel{\binom{13}{5}} \cdot 4^5}{\cancel{\binom{52}{5}}}} = \boxed{\left(\frac{3}{4}\right)^5} = 0.2373$$